R. G. W. H. STONE

PLANE AND SPHERICAL TRIGONOMETRY

GOODWIN.

TRIGONOMETRY

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TRIGONOMETRY

IN THREE PARTS

BY

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PREFACE

TO

THE FOURTH EDITION.

As indicated in the original Preface, this treatise in the first instance was intended to serve as an introduction to the study of Navigation and Nautical Astronomy for the junior officers under training in H.M. Fleet.

Since, however, it has had the good fortune to secure a somewhat more extended circulation, the Author takes advantage of the production of the fourth edition to largely supplement the number of examples, both theoretical and practical, so that while more fully meeting the requirements of naval students, the work may at the same time be rendered more complete in itself, and therefore more available for general purposes.

The new examples, to the number of nearly three hundred and fifty, will be found in an Appendix at the end of the volume. They have been selected from the papers set in examinations held under the direction of the Royal Naval College during recent years, and will, it is hoped, afford a sufficient field of exercise for the student in all branches of the subject.

ROYAL NAVAL COLLEGE:
September 1893.

PREFACE.

The following pages have been compiled chiefly for the use of the junior officers of H.M. Fleet, in whose studies the subjects of Plane and Spherical Trigonometry, forming, as they do, the basis of the sciences of Navigation and Nautical Astronomy, must necessarily occupy a very important place.

Since the establishment of the Royal Naval College at Greenwich a considerable advance has been made in the standard of mathematical knowledge attained by the junior officers of the Fleet, and for some time the need of a suitable treatise upon Plane and Spherical Trigonometry has been making itself more and more apparent.

The text-books in Trigonometry commonly used in the Service of late years have been four in number, viz. Hamblin Smith's Plane Trigonometry, Todhunter's Spherical Trigonometry, Johnson's Trigonometry (used on board H.M.S. *Britannia*), and Jeans' Trigonometry (used chiefly afloat).

The inconvenience which must attend the use of so varied a list of text-books is obvious, and, to remedy this drawback, in the year 1884 the Lords Commissioners of the Admiralty were pleased to give their approval to the preparation of the present work.

The Author has endeavoured to include within the compass of a single volume as much of the more theoretical portions of Plane and Spherical Trigonometry as is required in the final

examination of acting sub-lieutenants at Greenwich, and at the same time not to lose sight of the special character which must belong to a work intended for naval students, in whose case the practical application of the logarithmic formulæ must necessarily be of paramount importance.

The book is divided into three parts, the third of which is devoted to the practical application of the various formulæ established in Parts I. and II.

In Part I., dealing with the theoretical portion of Plane Trigonometry, the ground covered is practically identical with the subject matter of the well-known manual of Hamblin Smith,—a work which has during the last ten years proved of great value as an elementary text-book.

Part II. contains as much of the theory of Spherical Trigonometry as is necessary to establish the various relations required in the solution of spherical triangles. This is a subject which has generally been found to present special difficulties to the young officer, because, on account of the early age at which he is compelled to give it his attention, he enters upon its study with a much smaller amount of mathematical knowledge than is possessed by those who take it up simply as a branch of their general education. An effort has therefore been made to exhibit the subject in its simplest form, and the chief purpose of its study by naval officers, viz. to serve as an introduction to the subject of Nautical Astronomy, has been kept steadily in view.

Part III., the practical portion of the work, consists, to a great extent, of the examples in the use of logarithms and in the solution of plane and spherical triangles, compiled by the late Mr. H. W. Jeans, formerly Mathematical Master at the Royal Naval College at Portsmouth. Jeans' Trigonometry has been in constant use in the Royal Navy for many years, and there seems reason to believe that the collection of examples given in that book has been found to answer satisfactorily the purposes for which it was intended.

The miscellaneous examples given at the end of Part III.. as exercises in Practical Spherical Trigonometry, may perhaps be considered to belong rather to the sciences of Navigation and Nautical Astronomy; but, as no collection of such examples is to be found in any of the text-books in ordinary use afloat, the importance of these problems appears to justify their introduction here.

It will be observed that in the practical solution of triangles the cumbrous verbal rules which, in the darker days of naval education, were considered necessary, have been discarded, and the particular process of computation has been deduced directly from the appropriate formula in each case.

In obtaining the answers to the various practical problems the ordinary custom has been followed of looking out logarithms for the value given in the Tables which is nearest to the given angle, that is, in general, to the nearest 15″.

The Author wishes to take this opportunity of thanking the several friends who have been good enough to assist him with criticism and suggestions. To the Rev. J. B. Harbord, Chaplain of the Fleet, the Rev. J. C. P. Aldous and the Rev. S. Kenah, of H.M.S. *Britannia*, and the Rev. J. L. Robinson, of the Royal Naval College, his acknowledgments are especially due.

Royal Naval College, Greenwich:
March 1886.

CONTENTS.

PART I.

PLANE TRIGONOMETRY.

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PART II.

SPHERICAL TRIGONOMETRY.

PART III.

PRACTICAL TRIGONOMETRY.

PLANE AND SPHERICAL.

APPENDIX.

PART I.

PLANE TRIGONOMETRY

CHAPTER I.

ON MEASUREMENT, UNIT, RATIO.

1. In a subject dealing with concrete quantities, it is necessary to fix upon standards of measurement, by reference to which we may form definite conceptions of the magnitude of those quantities. Thus, in such expressions as 'a journey of fifty miles,' 'a reign of thirty years,' 'a legacy of one thousand pounds,' our notions of the distance, time, and value involved are derived from the several standards—one mile, one year, and one pound respectively. We consider how many times each of the fixed measures is contained in the aggregate quantity which we have in view, and thus arrive at its numerical measure.

2. And since each standard measure contains itself once, its measure is therefore represented by unity, and with reference to that particular system of measurement it is termed a *unit.*

3. In trigonometry the systems of measurement with which we have to deal relate, firstly, to lines, and secondly, to angles.

4. To measure a line AB, we fix upon a line of given length as a standard of linear measurement. Thus, if AB contain the line p times, p is called the measure of AB, and AB is represented algebraically by the symbol p.

5. Two lines are called *commensurable* when such a line, or unit of length, can be found that it is contained an exact number of times in each. When this is not the case, the lines are said to be *incommensurable*, and the comparison of their lengths can only be expressed approximately by figures.

Example 1.—What is the measure of $1\frac{1}{2}$ miles, when a length of 4 feet is taken as the unit?

$$1\tfrac{1}{2} \text{ miles} = \frac{3}{2} \times 1760 \times 3 \text{ feet} = 7920 \text{ feet} = 1980 \times 4 \text{ feet}.$$

Therefore the measure of $1\frac{1}{2}$ miles is 1980.

Example 2.—If 360 square feet be represented by the number 160, what is the unit of linear measurement?

The unit of area, that is, the square area which has for its side the unit of length, $= \frac{360}{160}$ square feet $= \frac{9}{4}$ square feet.

Therefore the side of this area $= \sqrt{\frac{9}{4}}$ linear feet $= \frac{3}{2}$ linear feet, or 18 inches.

Example 3.—If $4\frac{1}{2}$ inches be the unit of length, find the volume of a block of wood which is represented by the number 32.

The unit of volume, viz., the cube which has for its edge the unit of length $= \frac{9}{2} \times \frac{9}{2} \times \frac{9}{2} = \frac{729}{8}$ cubic inches.

Therefore the volume of the cube represented by $32 = \frac{729}{8} \times 32$ cubic inches $= 2916$ cubic inches.

Examples.—I.

1. If 5 inches be the unit of length, by what number will 6 yards 4 inches be represented?

2. If 3 furlongs be represented by the number 11, what is the unit?

3. Find the measure of an acre when 11 yards is the unit of linear measurement.

4. Find the measure of a yards when the unit is b feet.

5. A line referred to different units has measures 7 and 3; the first unit is 9 inches: what is the other?

6. A block of stone containing 3430 cubic inches is represented by the number 270; what is the unit of length?

6. Thus it will be seen that in order to obtain the measure of a given straight line, with reference to a given unit, we in reality form a fraction, which has for its numerator the length of the given straight line, and for its denominator the length of the given unit; a fraction which, when reduced to its lowest terms, is not necessarily a proper fraction, but is sometimes a whole number, and sometimes an improper fraction.

The fraction so formed is said to represent the *ratio* of the given straight line to the given unit.

7. To ask, therefore, 'What is the ratio of 3 yards to 7 inches?' is the same thing as to ask, 'What is the measure of 3 yards when 7 inches is taken as the unit?' or, 'What fraction is 3 yards of 7 inches?'

Generally, then, to express the ratio of one straight line to another, we have but to reduce the lengths of the lines to the same denomination; then taking one of these values for numerator, and the other for denominator, we obtain a fraction which expresses the ratio of the lengths of the two lines.

Thus, if we have two lines, AB 3 inches in length, and CD $1\frac{1}{4}$ feet, the ratio of AB to CD is expressed by the fraction $\frac{3}{15}$, or $\frac{1}{5}$. Similarly the ratio of CD to AB is represented by the number 5.

8. Let us suppose that, in accordance with the system of measurement which has been described, the values of the three sides of a right-angled triangle are represented when referred to a common unit of measurement by the letters p, q, r, the last of these being opposite to the right angle.

The theorem established by Euclid, I. 47, furnishes us with an equation—

$$p^2 + q^2 = r^2,$$

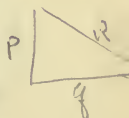
from which, if any two of these quantities be known, we may determine the third. Thus, if $p = 6$, and $r = 10$, we have—

$$36 + q^2 = 100;$$
$$\therefore q^2 = 64, \text{ and } q = 8.$$

Examples.—II.

1. The ratio of the heights of two walls is as 7 to 5; the height of the second one is 8 ft. 4 in.: what is the height of the first?

2. A field containing one acre is represented in one system of measurement by $10m^2$; in another system a furlong is de-

noted by $20m$; find the ratio of the linear units made use of in the two cases.

3. The length of the hypothenuse of a right-angled triangle is 80 feet, and one of the sides including the right angle is 64 feet; find the third side.

4. Find approximately the diagonal of a rectangle, the sides of which are 7 feet and 5 feet respectively.

5. ABC is an isosceles triangle, each of the equal sides being 20 feet; the length of the line which bisects the vertical angle is 16 feet; find the third side of the triangle.

6. The area of a square is 12,769 yards; find approximately the length of its diagonal.

7. If in an equilateral triangle the length of the perpendicular let fall from an angular point upon the opposite side be a feet, what is the length of the side of the triangle?

8. A square is inscribed in a circle; find the ratio of the side of the square to the radius of the circle.

9. The length of the chord of a circle is 8 feet, and its distance from the centre is 3 feet; find the length of the diameter.

10. The radius of a circle is 1 foot; find approximately the length of a chord which is 2 inches distant from the centre.

CHAPTER II.

ON THE MEASUREMENT OF ANGLES.

9. THE student has been accustomed in geometry to the definition of a plane rectilineal angle as 'the inclination of two straight lines to one another, which meet, but are not in the same straight line'; a definition which restricts the magnitude of the angle to angles less than two right angles. In trigonometry this definition is extended, and the conception of an angle is derived from the revolution of a straight line round one end of it, which remains fixed.

Thus, let AOB, COD be two straight lines intersecting at O,

and at right angles to each other, and let OP be a line free to revolve round the point O.

The trigonometrical angle is measured by the angle moved through by this revolving line.

Thus let the line be supposed at first to coincide with OA, and to move from right to left, or in a direction contrary to that of the hands of a watch.

We have in the figure four positions of the revolving line, and the angle traced out, AOP, lies in the several cases within the following limits :—

(1) AOP is less than one right angle.

(2) AOP is less than two right angles, but greater than one right angle.

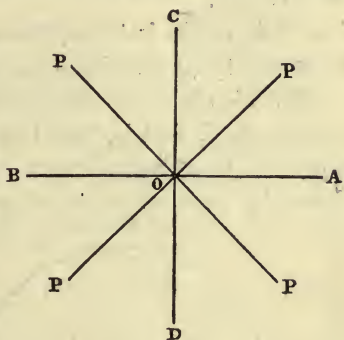
(3) AOP is less than three, but greater than two right angles.

(4) AOP is less than four, but greater than three right angles.

Since, after reaching the initial position OA, the revolution of the line OP may be continued indefinitely, there are evidently no limits to the magnitude of the trigonometrical angle.

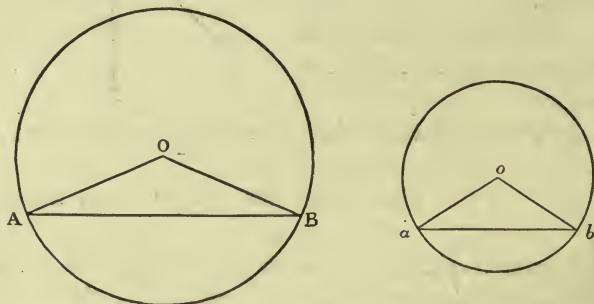
10. It becomes necessary, as in the case of straight lines, to adopt suitable units of measurement, which will enable us to form an idea of the magnitude of a given angle.

11. The most obvious method which suggests itself is to take for our unit a right angle, or some fraction of a right angle. Accordingly in what is known as the sexagesimal system a right angle is divided into 90 parts, each of which is called a degree, and the degree is subdivided into 60 smaller parts called minutes, while the minute also is divided into 60 seconds. Degrees, minutes, and seconds are then represented respectively by the symbols °, ′, ″, so that an angle is written as follows: 76° 18′ 25″.

12. We need not, however, confine ourselves to the right

angle as the primary unit. Any angle whose value is invariable will answer the purpose.

The method of measuring angles now to be described is known as the system of 'circular measurement,' and the unit to which all angles are referred is the angle which at the centre of any circle is subtended by an arc equal in length to the radius of the circle.

If this angle is to be taken as the unit, its magnitude must be shown to be invariable, whatever the size of the circle.

13. We shall first show that *the circumferences of circles vary as their radii*, that is, that if the radius of a circle be given, the length of the circumference may be found by multiplying the radius by a certain constant quantity.

Let O and o be the centres of two circles.

Let AB and ab be the sides of regular polygons of n sides inscribed in the circles. Let P be the perimeter of the polygon inscribed in the first circle, p the perimeter of that inscribed in the second circle, and let C, c be the circumferences of the two circles respectively.

Then the angles AOB, aob, being each equal to $\frac{1}{n}$th part of four right angles, are equal to one another.

Hence it is evident that the two triangles are equiangular. Therefore their sides about the equal angles are proportionals.

$$\text{Therefore} \quad \text{OA} : oa :: \text{AB} : ab;$$
$$:: n \,.\, \text{AB} : n \,.\, ab;$$
$$:: \text{P} : p.$$

But when n is very large, the perimeters of the polygons may be regarded as equal to the circumferences of the circles.

Therefore $\quad OA : oa :: C : c;$

or alternately $\quad OA : C :: oa : c.$

14. It is thus established that in any circle the circumference bears a constant ratio to the radius of the circle. The fraction, therefore, which represents the ratio, viz. $\dfrac{circumference}{radius}$ and therefore also the fraction $\dfrac{circumference}{diameter}$, is the same for all circles. The numerical value of this ratio is an incommensurable quantity, but it has been determined approximately by various methods. The fraction $\dfrac{22}{7}$ is a rough approximation; it is more nearly expressed by $\dfrac{355}{113}$, and its value to five places of decimals is 3·14159. The numerical value of this ratio is universally expressed by the symbol π.

The value of π may be roughly tested by actual measurement. Thus, if we take a circular hoop of radius 3·5 inches, and a piece of string be drawn round the hoop, its length will be found to be very nearly 22 inches.

15. Since $\dfrac{circumference}{diameter} = \pi$, it follows that

$$circumference = 2\pi r.$$

Therefore the arc of a quadrant $= \dfrac{1}{4}\, 2\pi r = \dfrac{\pi r}{2}$.

EXAMPLES.—III.

(In these examples the value of π may be taken as $\dfrac{22}{7}$.)

1. The diameter of a coin is $1\frac{1}{4}$ inches; find the length of its circumference.

2. The wheel of a bicycle makes 110 revolutions in a quarter of a mile; find the diameter of the wheel.

3. The minute hand of a watch is ·6 inch in length; in what time will the extremity travel one inch?

4. How many inches of wire would be necessary to make a

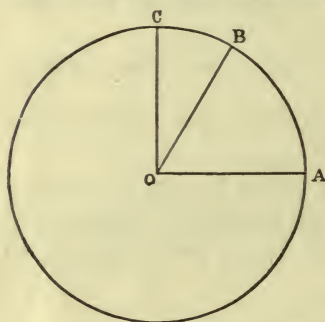
figure consisting of a circle of diameter one inch, with a square inscribed in it?

5. The barrel of a rifle is 1·75 inches in circumference; find the radius of a spherical bullet which will just fit the barrel.

6. A person who can run a mile in 5 minutes runs round a circular field in $1^m\ 25^s$; find the diameter of the field.

16. *To show that the angle subtended at the centre of a circle by an arc equal in length to the radius is an invariable angle.*

Let O be the centre of the circle whose radius $OA = r$.

At O in the straight line AO make the angle AOC a right angle, and let AB be an arc equal in length to the radius of the circle.

Then since, by Euclid, VI. 33, angles at the centre of a circle have to one another the same ratio as the arcs upon which they stand, we have—

$$\frac{\text{angle AOB}}{\text{angle AOC}} = \frac{\text{arc AB}}{\text{arc AC}} = \frac{r}{\frac{\pi r}{2}}.$$

Therefore the angle

$$\text{AOB} = \frac{2}{\pi}\text{ angle AOC, or} = \frac{\text{two right angles}}{\pi}.$$

Now π, as we have seen, is a quantity whose value does not change.

Therefore the angle AOB has always the same magnitude, whatever be the value of the radius OA, so that this angle may properly be adopted as the unit in a system of measurement.

17. *To show that the circular measure of an angle is the ratio of the arc on which it stands to the radius of the circle.*

Thus, let AB be an arc equal to r, the radius of the circle. Let AC be an arc, of length l, subtending an angle AOC at the centre.

Then, by Euclid, VI. 33,

$$\frac{\text{angle AOC}}{\text{angle AOB}}=\frac{\text{arc AC}}{\text{arc AB}}=\frac{l}{r}.$$

Therefore $\text{AOC}=\frac{l}{r}\ \text{AOB}=\frac{l}{r}$, since AOB is unity.

The fraction $\frac{l}{r}$ is called the 'circular measure' of the angle AOC, so that 'circular measure' may be defined as follows:—

The circular measure of an angle is the fraction which has for its numerator the length of the arc subtended by the angle at the centre of any circle, and for its denominator the radius of the same circle.

In the case of a right angle, since, as has been shown, the arc of a quadrant is $\frac{\pi r}{2}$, the fraction $\frac{l}{r}$ becomes $\frac{\frac{\pi r}{2}}{r}$, or $\frac{\pi}{2}$.

The circular measure of a right angle is therefore $\frac{\pi}{2}$; so that the circular measure of two right angles, or 180°, is π.

18. We shall proceed to investigate methods of converting the measure of an angle from one system to the other.

Thus let D be the number of degrees in an angle; it is required to find its circular measure x.

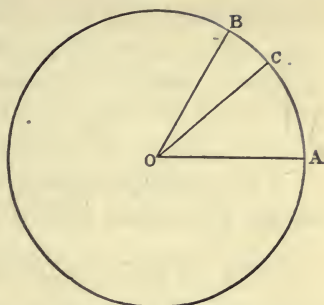
Since π is the circular measure of 180°, and since, whatever be the system of measurement, the values which represent two angles must have the same ratio to one another in each system, we obtain the equation—

$$\frac{x}{\pi}=\frac{\text{D}}{180};\ \text{therefore}\ x=\frac{\text{D}}{180}\pi\quad(1).$$

Again, when θ, the circular measure, is given, and it is required to find the number of degrees.

Let y be the number required; then

$$\frac{y}{180}=\frac{\theta}{\pi};\ \text{therefore}\ y=\frac{\theta}{\pi}\,180\quad(2).$$

To obtain the number of degrees in the unit of circular

measure, *i.e.* in the angle which at the centre of any circle subtends an arc equal in length to the radius, we have only to substitute unity for θ in equation (2).

$$\text{Thus, } y = \frac{180^\circ}{\pi} = 57^\circ{\cdot}29577 \ . \ . \ .$$

If we remember this value, we can at once realise in degrees, &c., the value of an angle given in circular measure.

Thus, suppose the circular measure to be $\frac{1}{2}$; its value in degrees is $\frac{1}{2}(57^\circ{\cdot}29577 \ . \ . \ .)$, or $28^\circ{\cdot}64788$ . . . and so on.

EXAMPLES.—IV.

1. Express in degrees, &c., the angles whose circular measures are

$$(1)\ \frac{1}{3};\quad (2)\ 1\frac{3}{4};\quad (3)\ \frac{11}{16};\quad (4)\ \frac{\pi}{3};\quad (5)\ \frac{\pi}{8};\quad (6)\ \frac{17}{16}\pi.$$

2. Express in circular measure the following angles :—

(1) 30° (2) 75° (3) 420°

(4) the angle of an equilateral triangle.

3. If the unit of angular measurement be one-sixth part of a right angle, by what number would an angle of 100° be expressed ?

4. What must be the unit of measurement when three-fourths of a right angle is expressed by the number 9 ?

5. An isosceles triangle has each of the angles at the base four times the vertical angle; express each of the angles in circular measure.

6. If a right angle be taken as the unit of measurement, what is the measure of the angle which, at the centre of a circle, subtends an arc equal in length to the radius ?

7. The angles of a triangle are in arithmetical progression; show that one of them must be 60°.

8. If the unit be the angle subtended at the centre of a circle by an arc equal in length to twice the radius, by what number would three right angles be represented ?

9. One angle of a triangle is 45°, and the circular measure of another is 1·5 ; find the circular measure of the third angle.

10. The angles of a triangle, in ascending order of magni-

tude, if expressed in terms of 1°, 100′, and 200′ respectively as units, are numerically equal; find the angles.

11. The angles of a plane triangle are in arithmetical progression; if the circular measure of the smallest be equal to $\frac{1}{100}$th part of the number of the degrees of the angle next in size, find the circular measure of the greatest angle.

12. Express in circular measure the angle of a regular hexagon.

13. If ABCD . . . be a regular decagon, and the sides AB, DC be produced to meet, express in degrees and in circular measure the angle contained by the pair of sides so produced.

14. In a certain circle an angle of 30° at the centre subtends an arc three feet in length; find the length of the radius.

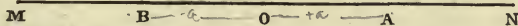
15. The measure of a certain angle in degrees exceeds its circular measure by unity; find the value of the angle in degrees.

16. The angle subtended by the moon's diameter is 32′ 30″; find approximately its length in miles, if the distance from the observer be 240,000 miles.

CHAPTER III.

ON THE APPLICATION OF ALGEBRAICAL SIGNS.

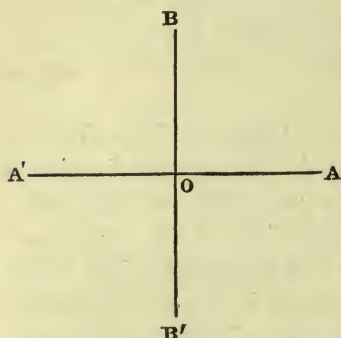
19. Let MN be a straight line, and O a fixed point in it.

M B O A N

The position of any point in the line will be determined if we know the distance of the point from O, and if we know also upon which side of O the point lies. It is, therefore, convenient to bring into requisition the algebraical signs + and −, so that if distances measured along the straight line from the point O in one direction be considered positive, distances measured along the straight line in the opposite direction will be considered negative.

For instance, let A be a point to the right of O, distant from it a linear units. Then if B be a point to the left of O, and equally distant from it, the distance of B from O is properly expressed by $-a$.

It should be observed that in the first instance either direc-

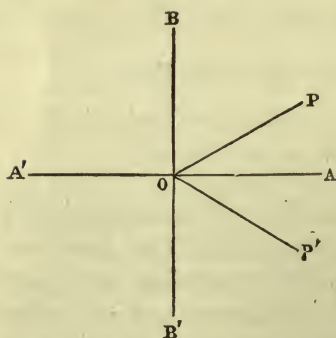
tion may be selected as the positive direction, provided that when once made the selection is rigidly adhered to afterwards.

20. Let AOA′, BOB′ be two straight lines intersecting at right angles in O.

Then if lines measured along OA, OB, be regarded as positive, lines measured along OA′, OB′, must be considered negative.

This *convention*, as it is called, is extended to lines parallel to AA′ and BB′ as follows:—

Lines drawn parallel to AA′ are considered positive when measured to the right of BB′, and negative when to the left of BB′.

Lines drawn parallel to BB′ are reckoned positive when measured upwards from AA′, and negative when measured downwards from AA′.

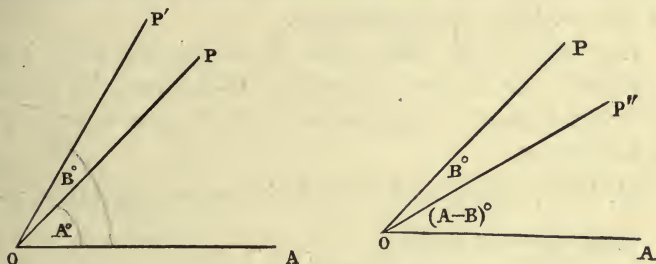
21. We have seen in art. 9 that the angle in trigonometry is generated by the revolution of a straight line round one extremity, which remains fixed.

Let OP be the revolving line, and let us suppose that it is in the initial position, coinciding with OA.

It is obvious that the straight line may trace out angles in two ways, according as it moves in the direction of, or contrary to, the hands of a watch.

Let us therefore suppose the revolving line to have left the initial position, and, moving contrary to the hands of a watch, to have reached the position OP.

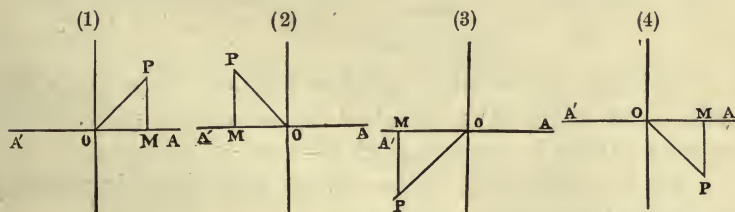
It is the general custom to call this the *positive* direction.

But if the revolving line be supposed to move downwards from OA until it reaches the position OP′, moving in the direction of the hands of a watch, the angle AOP′ and all other angles so traced out are considered negative angles.

22. Let the revolving line, moving in the positive direction, come to rest in the position OP, having traced an angle of A°. Afterwards let us suppose the revolving line to move through another angle of B°, first in the positive, then in the negative direction, finally arriving at the positions OP′, OP″ in the two cases.

Then the angles AOP′, AOP″ are respectively (A + B)°, and (A − B)°, these being the angles moved through by the revolving line in the two cases.

CHAPTER IV.

ON THE TRIGONOMETRICAL RATIOS.

23. Let the line OP, leaving the initial position OA, and revolving round O in the positive direction, describe an angle AOP.

From P drop a perpendicular upon OA, the initial line, or upon OA produced.

In fig. (1) the angle described is an acute angle, and is less than one right angle.

In fig. (2) it is greater than one right angle, but less than two right angles.

In fig. (3) it is greater than two, but less than three right angles.

In fig. (4) it is greater than three, but less than four right angles.

In each case our construction gives us a right-angled triangle POM, which is called the *triangle of reference.*

And in this triangle the side PM is the perpendicular, OM the base, and OP the hypothenuse.

The ratios which these parts of the triangle POM bear to one another are of great importance in dealing with the angle AOP, and to each of these ratios accordingly a separate name has been given.

24. The principal ratios are six in number, as follows:—

In the triangle POM

$\frac{PM}{OP}$, or $\frac{\text{perpendicular}}{\text{hypothenuse}}$, is the *sine* of AOP.

$\frac{OM}{OP}$, or $\frac{\text{base}}{\text{hypothenuse}}$, is the *cosine* of AOP.

$\frac{PM}{OM}$, or $\frac{\text{perpendicular}}{\text{base}}$, is the *tangent* of AOP.

$\frac{OP}{PM}$, or $\frac{\text{hypothenuse}}{\text{perpendicular}}$, is the *cosecant* of AOP.

$\frac{OP}{OM}$, or $\frac{\text{hypothenuse}}{\text{base}}$, is the *secant* of AOP.

$\frac{OM}{PM}$, or $\frac{\text{base}}{\text{perpendicular}}$, is the *cotangent* of AOP.

To these may be added another function of the angle, viz. the *versine*, which is the defect of the cosine from unity. Thus versine AOP $= 1 -$ cosine AOP.

25. The words sine, cosine, &c. are abbreviated in practice, the several ratios being written—

Sin. A, cos. A, tan. A, cosec. A, sec. A, cot. A, vers. A.

26. The powers of the various ratios are expressed in the following way:—

$$(\sin. A)^2 \text{ is written } \sin.^2 A;$$

and so on for the other ratios.

[The reader should carefully guard against confusing this symbol with sin. 2A, the meaning of which will appear further on.]

27. It should be noticed that the second three ratios defined above are respectively the reciprocals of the three first given. Thus—

$$\text{cosec. A} = \frac{1}{\text{sin. A}};$$

$$\text{sec. A} = \frac{1}{\text{cos. A}};$$

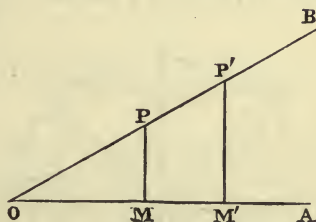
$$\text{cot. A} = \frac{1}{\text{tan. A}}.$$

28. *The trigonometrical ratios remain the same so long as the angle is unchanged.*

It has been explained that in Trigonometry an angle is supposed to be traced by the revolution of a straight line round one of its extremities.

Let AOB be any angle.

Take points P, P′ in OB, such that OP′ = m . OP.

Then we may suppose that either OP or OP′, originally coinciding with OA, has traced the angle AOB.

From P, P′ let fall perpendiculars PM, P′M′ upon OA.

Then the triangles POM, P′OM′ are equiangular, and therefore similar.

Hence the sides of the triangles about the equal angles are proportional.

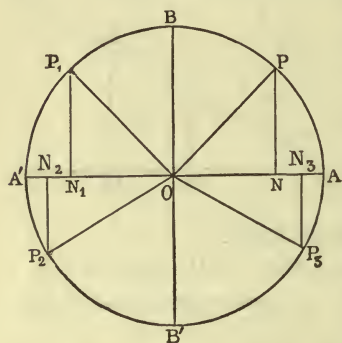
$$\text{And } \frac{\text{P}'\text{M}'}{\text{OP}'} = \frac{m\,.\,\text{PM}}{m\,.\,\text{OP}} = \frac{\text{PM}}{\text{OP}} = \text{sin. AOB}.$$

$$\text{So } \frac{\text{OM}'}{\text{OP}'} = \frac{m\,.\,\text{OM}}{m\,.\,\text{OP}} = \frac{\text{OM}}{\text{OP}} = \text{cos. AOB}.$$

And similarly for the other functions of the angle AOB. So that it is a matter of indifference whether we consider the angle AOB to belong to the triangle POM or to the triangle P′OM′. In each case the values obtained for the several ratios will be the same.

CHAPTER V.

TO TRACE THE CHANGES, IN SIGN AND MAGNITUDE, OF THE DIFFERENT TRIGONOMETRICAL RATIOS OF AN ANGLE, AS THE ANGLE INCREASES FROM 0° TO 360°.

29. LET AA′, BB′ bisect each other at right angles at the point O, and let a line OP revolve in the positive direction round the fixed point O, so that its extremity P traces the circumference of a circle.

Thus, as in art. 23, if P, P_1, P_2, P_3 be the positions of P in the first, second, third, and fourth quadrants respectively, by dropping perpendiculars PN, P_1N_1, P_2N_2, P_3N_3 upon the initial line, we obtain in each case a triangle of reference, from the sides of which the geometrical representations of the several ratios $\frac{PN}{OP}$, $\frac{ON}{OP}$, &c. may be derived.

And in accordance with the convention of the preceding chapter, the lines PN, P_1N_1, &c., will be considered positive so long as they are above the line AA′, and negative when below that line.

Similarly the lines ON, ON_1, &c. will be regarded as positive when drawn to the right of BB′, and negative when to the left of that line.

The line OP_1 is to be regarded as uniformly positive.

30. *To trace the changes in the sine of an angle as the angle increases from 0° to 360°.*

Let A be the angle traced by the revolving line.

$$\text{By art. 24, sin. } A = \frac{\text{perpendicular}}{\text{hypothenuse}} = \frac{PN}{OP}.$$

Now when $A = 0°$, OP coincides with OA, and the perpendicular PN vanishes.

$$\text{Therefore sin. } A = \frac{0}{OP} = 0.$$

When the angle therefore is zero, so also is its sine. As A increases from 0° to 90° PN is positive, and continually increases from 0 to OP.

Thus when $A = 90°$

$$\text{sin. } A = \frac{OP}{OP}.$$

Hence in the first quadrant sin. A is positive, and continually increases from 0 to 1.

As A increases from 90° to 180° the perpendicular PN remains positive, but decreases continually until when $A = 180°$ PN again vanishes, and sin. $180° = 0$. In the second quadrant therefore sin. A is positive, and decreases from 1 to 0.

As A increases from 180° to 270° the perpendicular PN becomes negative in sign, but increases until on reaching 270° it coincides with OB′, so that sin. $270° = -1$.

Hence in the third quadrant sin. A is negative, and increases numerically from 0 to -1.

As A increases from 270° to 360° PN remains negative in sign, and decreases, till on reaching the initial line it again vanishes.

Thus in the fourth quadrant sin. A is negative, and decreases numerically from -1 to 0.

31. *To trace the changes in the sign and magnitude of the cosine of an angle as the angle increases from* 0° *to* 360°.

The same construction being made, we have

$$\text{Cos. } A = \frac{\text{base}}{\text{hypothenuse}} = \frac{ON}{OP}.$$

Proceeding in the manner of the preceding article, we have, when the angle is 0°, OP coinciding with OA.

Hence $$\cos. 0° = \frac{OA}{OA} = 1.$$

As the angle increases from 0° to 90°, ON is positive, and continually decreases until, when OP reaches OB, ON vanishes altogether. Hence cos. 90° = 0.

Therefore in the first quadrant the cosine is positive, and decreases from 1 to 0.

By the same process the following results may easily be established:—

In the second quadrant cos. A is negative, and increases numerically from 0 to −1.

In the third quadrant cos. A is negative, and decreases numerically from −1 to 0.

In the fourth quadrant cos. A is positive, and increases from 0 to 1.

32. *To trace the changes in the sign and magnitude of the tangent as the angle increases from* 0° *to* 360°.

It should be observed that in selecting the proper sign to be affixed to the sine or cosine of a particular angle, we have only to consider what sign is due to the numerator of the fraction in each instance. With regard to the tangent and cotangent the case is otherwise, for with these ratios the sign belonging to both the numerator and denominator has to be taken into account.

When the angle is 0°, PN vanishes as before, while ON is equal to OA.

Therefore $\tan. A = \frac{PN}{ON} = \frac{0}{OA}$.

Therefore tan. 0° = 0.

As A increases PN, the numerator, is positive in sign, and increases until it ultimately coincides with OB. But ON, which was originally equal to OA, decreases continually, and eventually vanishes; so that since the numerator continually increases, while the denominator decreases, the value of the fraction continues to increase until upon the angle reaching 90°

$$\text{Tan. } A = \frac{OB}{0}.$$

Hence by taking the angle sufficiently near 90° the tangent may be made greater than any assigned value.

This is expressed by saying that tan. 90° is infinity, which is denoted by the symbol ∞.[1]

And between 0° and 90°, ON and PN being both positive, tan. A, or $\frac{PN}{ON}$, is also positive.

Thus in the first quadrant tan. A is positive, and increases from 0 to ∞.

As A increases from 90° to 180°, PN continues positive, and decreases, until at 180° it vanishes. ON is negative, and increases until it coincides with OA′.

Therefore in the second quadrant tan. A is negative, and decreases from ∞ to 0.

As A increases from 180° to 270°, PN is negative, and increases until ultimately it coincides with OB′; ON also is negative, and decreases until at 270° it vanishes.

The tangent is therefore positive in sign, and increases from 0 to ∞.

Between 270° and 360°, PN is negative, and decreases; ON is positive, and increases.

The tangent is therefore negative, and decreases numerically from ∞ to 0.

33. Since by art. 27 cosec. $A = \frac{1}{\text{sin. } A}$, sec. $A = \frac{1}{\text{cos. } A}$ cot. $A = \frac{1}{\text{tan. } A}$, the changes in sign and magnitude of the other ratios may be deduced from those already investigated, or they may be obtained directly from the figure. To trace them for himself will be found a useful exercise for the student.

[1] This result may be illustrated as follows. Suppose ON $= \frac{1}{n}$ OB, then tan. $A = \frac{OB}{\frac{1}{n}OB} = n$. Now as ON diminishes the quantity n increases, so that when ON becomes indefinitely small, tan. A or n becomes indefinitely large. That is, an angle may be found approximately equal to 90° whose tangent is greater than any assignable quantity.

34. The following table exhibits the changes in the values of the several ratios in a convenient form.

The intermediate columns show the sign possessed by the particular ratio as the angle increases from one value to the next higher in magnitude :—

Value of A	0°		90°		180°		270°		360°
Sine . . .	0	+	1	+	0	–	–1	–	0
Cosine . .	1	+	0	–	–1	–	0	+	1
Tangent. .	0	+	∞	–	0	+	∞	–	0
Cosecant .	∞	+	1	+	∞	–	–1	–	∞
Secant . .	1	+	∞	–	–1	–	∞	+	1
Cotangent .	∞	+	0	–	∞	+	0	–	∞

35. The following points may be noticed in connection with the values assumed by the different ratios :—

The sine and cosine are never greater than unity.

The cosecant and secant are never less than unity.

The tangent and cotangent may have any values whatever from zero to infinity.

The trigonometrical ratios change sign in passing through zero or infinity, and through no other values.

CHAPTER VI.

ON THE RATIOS OF CERTAIN ANGLES IN THE FIRST QUADRANT.

36. We have seen that the trigonometrical ratios, the sine, cosine, &c. are simply numerical quantities. They can be found approximately for all angles by methods which cannot be at present explained. There are, however, certain angles the ratios of which can be determined in a simple manner. Among these are the angles 30°, 45°, and 60°.

37. *To find the trigonometrical ratios for an angle of* 30°.

Let OP, revolving from the position OA, describe an angle AOP, equal to one-third of a right angle, that is, an angle of 30°.

From P draw PM perpendicular to OA, and produce PM to meet the circle in P′. Join OP′.

Then the two triangles OPM, OP′M are equal in all respects. And the angle OP′M = OPM = 90° − AOP = 60°.

Thus the triangle OPP′ is equilateral.

Therefore $PM = \frac{1}{2} PP' = \frac{1}{2} OP$.

Let $2m$ be the measure of OP. Then m is the measure of PM. And $OM = \sqrt{4m^2 - m^2} = \sqrt{3m^2} = \sqrt{3} . m$.

$$\text{Then sin. } 30^\circ = \frac{PM}{OP} = \frac{m}{2m} = \frac{1}{2},$$

$$\text{cos. } 30^\circ = \frac{OM}{OP} = \frac{\sqrt{3} . m}{2 . m} = \frac{\sqrt{3}}{2};$$

$$\text{tan. } 30^\circ = \frac{PM}{OM} = \frac{m}{\sqrt{3} . m} = \frac{1}{\sqrt{3}};$$

whence also cosec. $30^\circ = 2$, sec. $30^\circ = \frac{2}{\sqrt{3}}$, cot. $30^\circ = \sqrt{3}$.

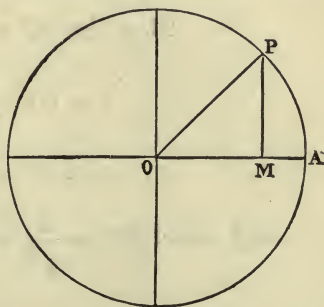
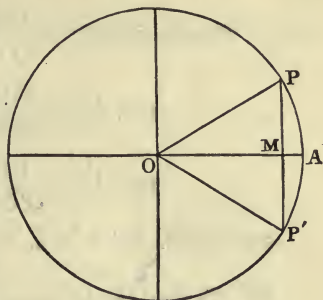
38. *To find the trigonometrical ratios for an angle of* 45°.

Let OP, revolving from the position OA, describe an angle AOP, equal to half a right angle, that is, an angle of 45°.

Draw PM perpendicular to OA.

Then, since POM, OPM are together equal to a right angle, and POM is half a right angle, therefore OPM also is half a right angle.

Thus OPM is equal to POM, and PM equal to OM.

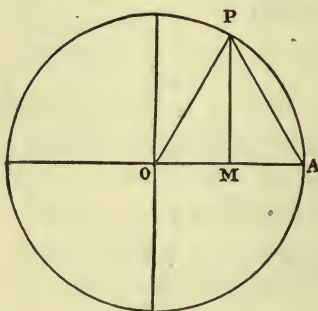
Let the measure of OM or PM be m.

Then the measure of OP is $\sqrt{m^2+m^2} = \sqrt{2m^2} = \sqrt{2}m$.

Thus $\sin. 45° = \frac{PM}{OP} = \frac{m}{\sqrt{2}m} = \frac{1}{\sqrt{2}}$;

$$\cos. 45° = \frac{OM}{OP} = \frac{m}{\sqrt{2}m} = \frac{1}{\sqrt{2}};$$

$$\tan. 45° = \frac{PM}{OM} = \frac{m}{m} = 1;$$

and $\operatorname{cosec}. 45° = \sqrt{2}$, $\sec. 45° = \sqrt{2}$, $\cot. 45° = 1$.

39. *To find the trigonometrical ratios for an angle of* 60°.

Let OP, revolving from the position OA, describe an angle equal to two-thirds of a right angle, that is, an angle of 60°.

Draw PM perpendicular to OA, and join AP.

Because OP is equal to OA, therefore the angle OAP is equal to the angle OPA.

But these two angles are together equal to 120°; therefore each is equal to 60°, and the triangle AOP is equilateral.

Hence OA is bisected in M.

Let the measure of OM be m.

Then the measure of OP is $2m$.

And the measure of PM is $\sqrt{4m^2-m^2} = \sqrt{3m^2} = \sqrt{3}m$.

$$\text{Then } \sin. 60° = \frac{PM}{OP} = \frac{\sqrt{3}m}{2m} = \frac{\sqrt{3}}{2};$$

$$\cos. 60° = \frac{OM}{OP} = \frac{m}{2m} = \frac{1}{2};$$

$$\tan. 60° = \frac{PM}{OM} = \frac{\sqrt{3}\,m}{m} = \sqrt{3};$$

and $\operatorname{cosec}. 60° = \frac{2}{\sqrt{3}}$, $\sec. 60° = 2$, $\cot. 60° = \frac{1}{\sqrt{3}}$.

EXAMPLES.—V.

If $A = 90°$, $B = 60°$, $C = 45°$, $D = 30°$, prove the following relations:—

(1) $\text{Sin.}^2 D + \cos.^2 D = 1$.

(2) $\text{Cos.}^2 B - \sin.^2 B = 1 - 2 \sin.^2 B$.

(3) $\text{Sec.}^2 B = 1 + \tan.^2 B$.

(4) $\text{Sin. } B \tan. D + \tan. C \sin. D = 1$.

(5) $2 \cos.^2 C + \tan.^2 B = \cot.^2 D \sec.^2 C - \text{cosec.}^2 C$.

(6) $\dfrac{\text{Sin. } C - \sin. D}{\text{Sin. } C + \sin. D} = (\sec. C - \tan. C)^2$.

(7) $\text{Sin. } A \sin. D - \cos. A \cos. C = 2 \text{ vers.}^2 B$.

(8) $\text{Tan.}^2 B - \tan.^2 D = \dfrac{\sin. A \sin. D}{\cos.^2 B \cos.^2 D}$.

CHAPTER VII.

ON THE RATIOS OF THE COMPLEMENT AND SUPPLEMENT.

40. *Def.* Two angles are said to be the complements of each other when their sum amounts to 90°.

Thus 60° is said to be the complement of 30°, and *vice versâ*; $-30°$ is the complement of 120°, $\frac{\pi}{6}$ of $\frac{\pi}{3}$, and so on.

41. *To compare the trigonometrical ratios of an angle and its complement.*

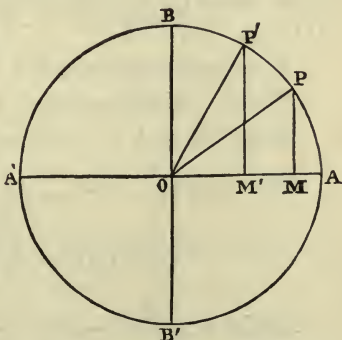
Let AA′, BB′ be two diameters of a circle intersecting at right angles in O.

Let a radius OP revolving from OA trace out the angle AOP = A.

Next let the radius revolve from OA to OB and back again through an angle BOP′ equal to A.

Then the angle AOP′ = 90° − A.

Draw PM, P′M′ perpendicular to OA.

Then the angle OP′M′ = BOP′ = A = MOP.

Therefore the triangles OPM, OP′M′, having one side and two angles equal, are equal in all respects; so that OM′ = PM, and P′M′ = OM.

Then

Sin. $(90° - A)$ = sin. $P'OM' = \frac{P'M'}{OP'} = \frac{OM}{OP}$ = cos. MOP = cos. A.

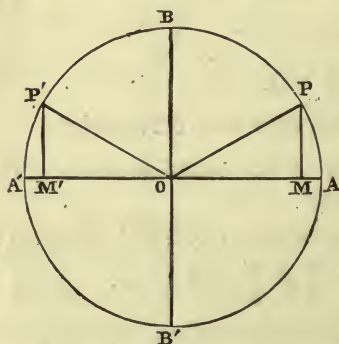
Cos. $(90° - A)$ = cos. $P'OM' = \frac{OM'}{OP'} = \frac{PM}{OP}$ = sin. MOP = sin. A.

And similarly it may be shown that tan. $(90° - A)$ = cot. A, cosec. $(90° - A)$ = sec. A, sec. $(90° - A)$ = cosec. A, cot. $(90° - A)$ = tan. A.

42. *Def.* Two angles are said to be the supplements of each other when their sum amounts to 180.°

Thus 150° is the supplement of 30°, $\frac{2\pi}{3}$ is the supplement of $\frac{\pi}{3}$.

To compare the trigonometrical ratios of an angle and its supplement.

Let AA′, BB′ be two diameters of a circle intersecting at right angles in O.

Let the line OP, revolving from OA, trace out an angle A.

Next let OP revolve from OA to OA′ and back through an angle equal to A, coming to rest in the position OP′, so that AOP′ = 180° − A.

Drop perpendiculars PM, P′M′ upon AA′.

The two triangles POM, P′OM′ are equal in all respects.

Thus PM = P′M′, OM = OM′.

$$\text{Therefore sin. } (180° - A) = \frac{P'M'}{OP'} = \frac{PM}{OP} = \text{sin. } A;$$

$$\text{cos. } (180° - A) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\text{ cos. } A.$$

Again tan. $(180° - A) = -$ tan. A; and in the same manner the other ratios may be compared.

43. *To compare the trigonometrical ratios of the angle* $(90° + A)$ *with those of* A.

Let AA′, BB′ be two diameters of a circle intersecting at right angles in O.

Let OP, revolving from OA, move to OP, describing an angle A.

Let OP then continue to revolve until after passing OB the angle BOP′ = A.

Then AOP′ = 90° + A.

Drop perpendiculars PM, P′M′ upon AA′.

In the triangle P′OM′ the angle OP′M′ = P′OB = A, and the two triangles POM, P′OM′ are equal in all respects.

Thus OM′ = PM, P′M′ = OM.

$$\text{Therefore} \quad \sin.\ (90° + A) = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cos.\ A\,;$$

$$\cos.\ (90° + A) = \frac{OM'}{OP'} = \frac{-PM}{OP} = -\sin.\ A.$$

Thus tan. (90° + A) = − cot. A, and similarly the other ratios may be compared.

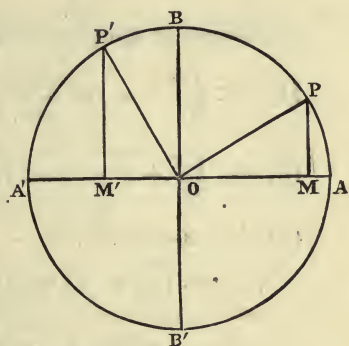
44. By processes similar to those already given the ratios of the angles (180° + A) and (− A) may be compared with those of the angle A, and will be found to be as follows:—

Sin. (180° + A) = − sin. A;	sin. (− A) = −sin. A;
cos. (180° + A) = − cos. A;	cos. (− A) = cos. A;
tan. (180° + A) = tan. A;	tan. (− A) = −tan. A.

EXAMPLES.—VI.

1. Write down the complements of the following angles:—

(1) 27° 37′ 48″. (2) 50° 16′ 38″. (3) 105°.

(4) − 37°. (5) $\frac{\pi}{3}$. (6) $-\frac{2\pi}{3}$.

2. Write down the supplements of the following angles ;—

(1) $79° \, 36' \, 15''$. (2) $101° \, 19' \, 43''$. (3) $200°$.

(4) $-70°$. (5) $\frac{2\pi}{5}$. (6) $-\frac{\pi}{7}$.

3. Reduce to simpler forms the following equations :—

(1) $\cos.\left(\frac{\pi}{2} - x\right) = \frac{\cos.(\pi - A)}{\sin.(-A)}$.

(2) $\sin.\left(\frac{\pi}{2} + x\right) = \cos.(\pi - A)\ \text{cosec.}(\pi + A)$.

(3) $\sin.(\pi - x)\cos.(\pi - B)$

$$= \frac{\cos.\left(\frac{\pi}{2} + A\right)\sec.(\pi + B)}{\text{cosec.}\left(\frac{\pi}{2} - A\right)}.$$

CHAPTER VIII.

ON THE RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS FOR THE SAME ANGLE.

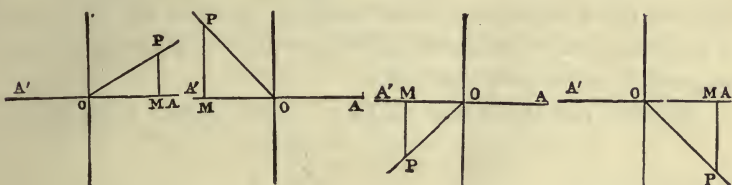
45. THE principal trigonometrical ratios, as defined in art. 24, are six in number, viz. the sine, cosine, tangent, cosecant, secant, and cotangent.

Three of these, as already pointed out, are the reciprocals of the other three ; thus—

$$\text{Cosec.}\,A = \frac{1}{\sin.A},\quad \sec.\,A = \frac{1}{\cos.A},\quad \cot.\,A = \frac{1}{\tan.A}.$$

By means of these relations, and others now to be established, any one of the ratios may be expressed in terms of each of the others.

46. Let AOP represent any angle A traced out by the revolution of OP round one extremity O, and let a perpendicular

PM be let fall upon OA, or OA produced, so that in each case POM is our triangle of reference.

We can now prove the following relations :—

I. $\tan. A = \frac{\sin. A}{\cos. A}$; $\cot. A = \frac{\cos. A}{\sin. A}$.

For $\tan. A = \frac{PM}{OM} = \frac{\frac{PM}{OP}}{\frac{OM}{OP}} = \frac{\sin. A}{\cos. A}$.

Again $\cot. A = \frac{OM}{PM} = \frac{\frac{OM}{OP}}{\frac{PM}{OP}} = \frac{\cos. A}{\sin. A}$.

II. $\text{Sin.}^2 A + \cos.^2 A = 1$.

For $\sin.^2 A + \cos.^2 A = \frac{PM^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{PM^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1$.

III. $\text{Sec.}^2 A = 1 + \tan.^2 A$; $\text{cosec.}^2 A = 1 + \cot.^2 A$.

For $\sec.^2 A = \frac{OP^2}{OM^2} = \frac{OM^2 + PM^2}{OM^2} = 1 + \frac{PM^2}{OM^2} = 1 + \tan.^2 A$.

$\text{Cosec.}^2 A = \frac{OP^2}{PM^2} = \frac{PM^2 + OM^2}{PM^2} = 1 + \frac{OM^2}{PM^2} = 1 + \cot.^2 A$.

Collecting our results, we know now that if A denote any angle, the following statements are true :—

(1) $\text{cosec. } A = \frac{1}{\sin. A}$; $\sec. A = \frac{1}{\cos. A}$; $\cot. A = \frac{1}{\tan. A}$.

(2) $\tan. A = \frac{\sin. A}{\cos. A}$; $\cot. A = \frac{\cos. A}{\sin. A}$.

(3) $\sin.^2 A + \cos.^2 A = 1$; $\sec.^2 A = 1 + \tan.^2 A$; $\text{cosec.}^2 A = 1 + \cot.^2 A$.

46a. We shall now give a few examples of what are known as 'identities.' An identity is an expression which states in the form of an equation certain relations between the functions of angles, these relations being true whatever magnitudes are assigned to the angles involved.

Example 1.—Prove that $\cot.^2\theta - \cos.^2\theta = \cot.^2\theta\cos^2\theta$.

$$\text{Since}\quad \cot.^2\theta = \frac{\cos.^2\theta}{\sin.^2\theta},$$

$$\cot.^2\theta - \cos.^2\theta = \frac{\cos.^2\theta}{\sin.^2\theta} - \cos.^2\theta = \frac{\cos.^2\theta - \cos.^2\theta\sin.^2\theta}{\sin.^2\theta}$$

$$= \frac{\cos.^2\theta(1 - \sin.^2\theta)}{\sin.^2\theta} = \frac{\cos.^2\theta}{\sin.^2\theta}\cos.^2\theta = \cot.^2\theta\cos.^2\theta.$$

Here our first step was to express $\cot.^2\theta$ in terms of the sine and cosine. This process of substituting for the tangent and other functions of an angle, their value in sines and cosines will be found of frequent advantage in such cases.

Example 2.—Prove that $\sin.^2\theta\tan.\theta + \cos.^2\theta\cot.\theta + 2\sin.\theta\cos.\theta = \sec.\theta\,\text{cosec}.\theta$.

$$\text{Since}\ \tan.\theta = \frac{\sin.\theta}{\cos.\theta},\ \cot.\theta = \frac{\cos.\theta}{\sin.\theta}.$$

$$\sin.^2\theta\tan.\theta + \cos.^2\theta\cot.\theta + 2\sin.\theta\cos.\theta = \frac{\sin.^3\theta}{\cos.\theta} + \frac{\cos.^3\theta}{\sin.\theta}$$
$$+ 2\sin.\theta\cos.\theta$$

$$= \frac{\sin.^4\theta + \cos.^4\theta + 2\sin.^2\theta\cos.^2\theta}{\sin.\theta\cos.\theta}$$

$$= \frac{(\sin.^2\theta + \cos.^2\theta)^2}{\sin.\theta\cos.\theta} = \frac{1}{\sin.\theta\cos.\theta} = \sec.\theta\,\text{cosec}\,\theta.$$

Examples.—VII.

Prove the identities—

(1) $\text{Sin}.A\sec.A = \tan.A$.

(2) $(\tan.A + \cot.A)\sin.A\cos.A = 1$.

(3) $(\tan.A - \cot.A)\sin.A\cos.A = \sin.^2A - \cos.^2A$.

(4) $(\cos.^4A - \sin.^4A) = 1 - 2\sin.^2A$.

(5) $\cos.^3A - \sin.^3A = (\cos.A - \sin.A)(1 + \sin.A\cos.A)$.

(6) $\cos.^6A + \sin.^6A = 1 - 3\cos.^2A\sin.^2A$.

(7) $2(\sin.^6A + \cos.^6A) - 3(\sin.^4A + \cos.^4A) + 1 = 0$.

(8) $(1 - 2\cos.^2 A)(\tan. A + \cot. A) =$
$(\sin. A - \cos. A)(\sec. A + \operatorname{cosec.} A)$.

(9) $\sec.^2 \theta - \cos.^2 \theta = \cos.^2 \theta \tan.^2 \theta + \sin.^2 \theta \sec.^2 \theta$.

(10) $(\operatorname{cosec.}^2 \theta - 1)(2\operatorname{vers.} \theta - \operatorname{vers.}^2 \theta) = \cos.^2 \theta$.

(11) $\sin. \theta (\cot. \theta + 2)(2 \cot. \theta + 1) = 2 \operatorname{cosec.} \theta + 5 \cos. \theta$.

(12) $\operatorname{vers.}^2 A + 2\cos. A - \sin.^2 A = 2\cos.^2 A$.

(13) $\operatorname{cosec.}^2 \theta - \operatorname{vers.} \theta = \operatorname{vers.} \theta \cot.^2 \theta + \cos. \theta \operatorname{cosec.}^2 \theta$.

(14) $\cos.^6 A + 2\cos.^4 A \sin.^2 A + \cos.^2 A \sin.^4 A + \sin.^2 A = 1$.

(15) $\dfrac{\tan. A + \tan. B}{\cot. A + \cot. B} = \tan. A \tan. B$.

(16) $\dfrac{\cot. A + \tan. B}{\tan. A + \cot. B} = \cot. A \tan. B$.

(17) $\dfrac{1 + \cos. A}{1 - \cos. A} = (\operatorname{cosec.} A + \cot. A)^2$.

(18) $\tan.^2 A - \tan.^2 B = \dfrac{\cos.^2 B - \cos.^2 A}{\cos.^2 A \cos.^2 B}$.

(19) $\sec. \theta + \operatorname{cosec.} \theta \tan.^3 \theta (1 + \operatorname{cosec.}^2 \theta) = 2 \sec.^3 \theta$.

(20) $2\operatorname{vers.} \theta - \operatorname{vers.}^2 \theta = \sin.^2 \theta$.

47. *To express all the other ratios in terms of the sine:*—

Since $\sin.^2 A + \cos.^2 A = 1$

$$\cos.^2 A = 1 - \sin.^2 A;$$

$$\therefore \cos. A = \pm\sqrt{1 - \sin.^2 A}.$$

Again $\tan. A = \dfrac{\sin. A}{\cos. A} = \dfrac{\sin. A}{\pm\sqrt{1 - \sin.^2 A}}$;

$$\operatorname{cosec.} A = \frac{1}{\sin. A};$$

$$\sec. A = \frac{1}{\cos. A} = \frac{1}{\pm\sqrt{1 - \sin.^2 A}};$$

$$\cot. A = \frac{\cos. A}{\sin. A} = \frac{\pm\sqrt{1 - \sin.^2 A}}{\sin. A}.$$

It will be observed that corresponding to a given value of the sine, there will be two values of the cosine, tangent, secant, and cotangent, since on the right-hand side of the equation either the positive or negative sign may be taken. This arises from

the fact that when the sine of an angle is given, more than one angle may be found which possesses this sine, although the cosine, tangent, &c. may not be the same for the two angles.

As a simple case, let sin. $A = \frac{1}{2}$.

From art. 37 we know that the sine given is that of 30°.

But since, by art. 42, sin. $(180° - A) =$ sin. A, the angle 150° also has this sine.

Thus there are two angles possessing the given sine, viz. 30° and 150°.

To find the cosines of these angles we have the formula—

$$\cos. A = \pm \sqrt{1 - \sin.^2 A} = \pm \frac{\sqrt{3}}{2}.$$

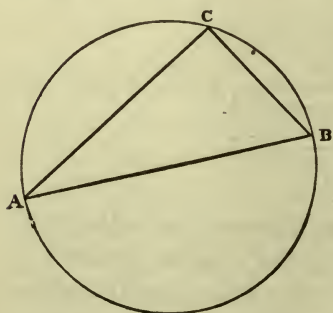
For 30°, since the angle lies in the first quadrant, the positive sign must be taken; for 150° the negative.

For the present the ambiguity in sign may, however, be neglected.

48. In a similar manner, by means of the formulæ given above, the other ratios may be expressed in terms of the cosine, tangent, &c. There is, however, another method of arriving at these expressions which is worthy of notice.

It will first be necessary to show how to construct an angle having a given sine, cosine, or tangent.

49. *To construct an angle having a given sine or cosine.*

Let it be required to construct the angle which has for its sine a given ratio a, a being less than unity.

Take AB equal to the unit of length, and upon AB as diameter describe a circle.

With centre B and radius BC, equal to the fraction a of the unit of length, describe a circle.

Let C be one of the points where the circumference of this circle intersects the first circle, and join AC, BC.

Then ACB, the angle in a semicircle, is a right angle (Euc. III. 31).

Thus $\sin. BAC = \frac{BC}{AB} = \frac{a}{1} = a.$

Therefore BAC is the angle required.

If the *cosine* of the required angle is to be a, the same construction may be made, and ABC will be the angle required.

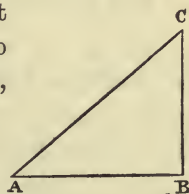
50. *To construct an angle having a given tangent or cotangent.*

Let it be required to construct an angle of which the tangent shall be a given quantity a.

Draw a straight line AB equal to the unit of length. At B draw BC at right angles to AB, and equal in length to a times the unit, and join AC.

C

A B

Then $\tan. BAC = \frac{BC}{AB} = a.$

Therefore BAC is such an angle as is required.

If the *cotangent* of the required angle is to be a, then the same construction may be made, and ACB will be such an angle as is required.

51. *To express the other trigonometrical ratios in terms of the sine.*

Let s be the sine of the given angle.

As in art. 49, construct a right-angled triangle having its hypothenuse equal to the unit of length, and the side BC s times the unit.

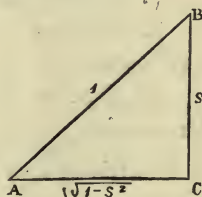
B

1 s

A $\sqrt{1-s^2}$ C

It follows that $AC = \sqrt{1-s^2}$.

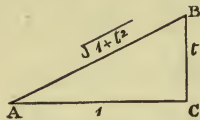
Then we have $\cos. A = \frac{AC}{AB} = \sqrt{1-s^2}$

$$\tan. A = \frac{BC}{AC} = \frac{s}{\sqrt{1-s^2}}.$$

Similar expressions will easily be found for the other ratios.

52. *To express the other trigonometrical ratios in terms of the tangent.*

Let t be the tangent of the angle.

As in art. 50, construct a right-angled triangle having the

side AC, one of those including the right angle, equal to the unit of length, and the other side BC t times the unit, so that BAC is the angle which has the given tangent.

Then $AB=\sqrt{1+t^2}$;

and $\sin. A = \frac{BC}{AB} = \frac{t}{\sqrt{1+t^2}}$;

$\cos. A = \frac{AC}{AB} = \frac{1}{\sqrt{1+t^2}}$.

And so for the other ratios.

53. In the case of the tangent the above method is the simplest for deducing expressions for the other ratios. We might, however, deduce the value of the sine, &c. in terms of the tangent by proceeding as follows :—

$\text{Tan. } A = \frac{\sin. A}{\cos. A}$;

Therefore $\tan.^2 A = \frac{\sin.^2 A}{\cos.^2 A} = \frac{\sin.^2 A}{1-\sin.^2 A}$.

$\therefore \tan.^2 A - \tan.^2 A \sin.^2 A = \sin.^2 A$.

Whence $\sin. A = \frac{\tan. A}{\sqrt{1+\tan.^2 A}}$.

In the same way the expression for the cosine might be obtained.

Examples.—VIII.

1. Express the sine of an angle in terms of the cotangent.

2. Express the sine and cotangent of an angle in terms of the secant.

3. Express the principal trigonometrical ratios in terms of the versine.

4. Given that $\cos. A = \frac{4}{5}$, find tan. A.

5. Given that sec. $A=2$, find cot. A.

6. If $\tan. \theta = \frac{2}{\sqrt{5}}$, find $\sin. \theta$ and $\sec. \theta$.

7. If $\sec. A = \frac{7}{5}$, find cot. A.

8. If sin. $A=\frac{1}{8}$, find tan. A.

9. If tan. $A=\frac{a}{b}$, find cosec. A.

10. If cos. $\theta=a$, and tan. $\theta=b$, find the equation connecting a and b.

11. Construct the angles whose sines are $\frac{1}{8}$ and $\frac{1}{\sqrt{3}}$.

12. Construct the angle whose tangent is $\sqrt{2}-1$.

CHAPTER IX.

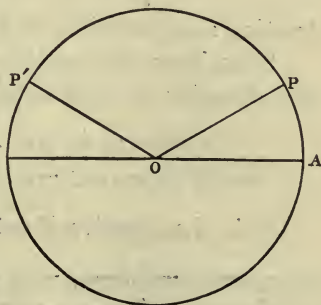
ON THE RATIOS OF ANGLES UNLIMITED IN MAGNITUDE.

54. It is shown in art. 41 that sin. $(180°-A)=$sin. A. It follows from this that if an angle be constructed in the first quadrant having its sine equal to a given positive quantity a, another angle may always be obtained in the second quadrant having its sine equal to the same quantity.

If the condition were that cos. A should be equal to a, since cos. $A=$cos. $(-A)$, or cos. $(360°-A)$, there would also be two values for the angle A, one in the first and the other in the fourth quadrant.

If tan. $A=a$, then the angle must be in the first or third quadrant.

55. Thus let sin. $A=\frac{1}{2}$. Here A may be either 30° or 150°, and the positions of the revolving line will be OP and OP′ respectively.

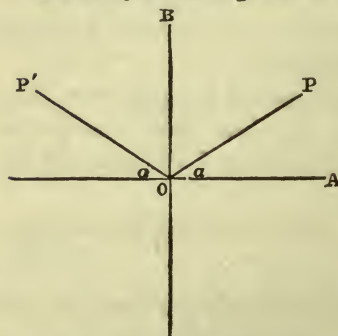
Moreover, since the trigonometrical angle is unlimited in magnitude, let us suppose the revolving line on reaching the initial position to continue its revolution; as often as it takes up the position OP or OP′ we shall obtain an angle having its sine equal to the assigned value, viz. $\frac{1}{2}$.

And OP of course is free to move either in the positive or negative direction.

Thus, if OP revolve from OA in the positive direction, we have a series of angles as follows: 30°, 150°, 390°, 510,° . . . all having the given value of the sine.

But if the negative direction be taken, we obtain a series of negative angles, viz. −210°, −330°, −570,°. . . also having the given sine.

Thus generally if a be the circular measure of the smallest angle having the given sine, by the addition or subtraction of 2π, 4π, 6π . . . $2n\pi$, we shall obtain a series of angles having their sine equal to the value given. And the same statement applies to the other ratios.

We shall proceed to find expressions which will include all angles having a given sine, cosine, or tangent.

56. *To find an expression which will include all angles having a given sine.*

Let a be the angle having a given sine.

The other position of the revolving line which gives the same value for the sine is that for the angle $\pi - a$.

All other angles positive and negative which have the same sine may be derived by adding or subtracting some multiple of 2π from these two values.

Thus if we suppose the revolving line to move in the positive direction we shall obtain the two series of angles

$$a,\ 2\pi + a,\ 4\pi + a \ . \ . \ . \ . \ . \ . \ 2m\pi + a$$
$$\pi - a,\ 3\pi - a,\ 5\pi - a \ . \ . \ . \ . \ . \ . \ (2m+1)\ \pi - a$$

If in the negative direction, we have

$$-\pi - a,\ -3\pi - a,\ -5\pi - a \ . \ . \ . \ . \ . \ -(2m+1)\ \pi - a$$
$$-2\pi + a,\ -4\pi + a,\ -6\pi + a \ . \ . \ . \ . \ . \ -2m\pi + a$$

where m may have any integral value.

Comparing these four series we see that

(1) π may have any integral coefficient, odd or even, positive or negative.

(2) When the coefficient of π is even the sign of a is $+$, when odd the sign of a is $-$.

Now both the conditions of (2) are satisfied by the expression

$$n\pi + (-1)^n a$$

where n is any integer, positive or negative.

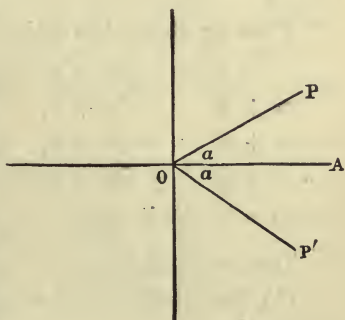
For if n be even $(-1)^n$ must be positive, but if n be odd $(-1)^n$ will be negative.

By assigning, therefore, in succession the numbers 1, 2, 3, 4, . . . and $-1, -2, -3, -4$, . . . to the letter n, we shall obtain the two series of angles which have the given sine.

57. *To find an expression which will include all angles having a given cosine.*

Let a be the angle having the given cosine.

The only other position of the revolving line which will give the same value for the cosine is that for the angle $2\pi - a$.

All other angles having the given cosine may be obtained by adding or subtracting 2π, and multiples of 2π, from each of these values.

The four series so obtained will be

$$\begin{array}{lllll} a, & 2\pi + a, & 4\pi + a & \ldots & 2m\pi + a \\ 2\pi - a, & 4\pi - a, & 6\pi - a & \ldots & 2m\pi - a \\ -2\pi + a, & -4\pi + a, & -6\pi + a & \ldots & -2m\pi + a \\ -a, & -2\pi - a, & -4\pi - a & \ldots & -2m\pi - a \end{array}$$

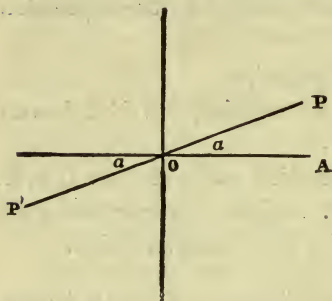
where m may have any integral value.

Here we observe

(1) That the coefficient of π is always even, but either positive or negative in sign.

(2) That a may have either a positive or negative sign.

Thus the expression $2n\pi \pm a$, where n is any integer, positive or negative, will furnish all values of the angle having the given cosine.

58. *To find an expression which will include all angles having a given tangent.*

Let a be the angle whose tangent is given.

The only other position of the revolving line for which the tangent will have the given value is that for the angle $\pi + a$. All other values of the angle having the given tangent will be obtained by adding or subtracting 2π, and multiples of 2π, from one of these angles.

Thus we obtain the series

$$\begin{array}{llll} a, & 2\pi + a, & 4\pi + a \ldots & 2m\pi + a \\ \pi + a, & 3\pi + a, & 5\pi + a \ldots & (2m + 1)\pi + a \\ -2\pi + a, & -4\pi + a, & -6\pi + a \ldots & -2m\pi + a \\ -\pi + a, & -3\pi + a, & -5\pi + a \ldots & -(2m + 1)\pi + a \end{array}$$

where m is any integer.

Thus we find that

(1) The coefficient of π may be either odd or even, positive or negative.

(2) The sign of a is always positive.

So that the expression $n\pi + a$, where n has any integral value, positive or negative, contains all the angles required.

EXAMPLES.—IX.

1. Write down the values of the following ratios :—

(1) tan. 225° (2) cos. (− 60°) (3) tan. 780°
(4) cot. 1035° (5) sec. 240° (6) cot. 210°
(7) cosec. 570° (8) sin. (−210°) (9) cos. (− 120°)
(10) tan. $\frac{10\pi}{3}$ (11) cot. $\frac{5\pi}{2}$ (12) tan. 6π

2. Find the general value of θ under the following circumstances:—

(1) When $\tan. \theta = 1$.

(2) When $\sin. \theta = \frac{1}{2}$.

(3) When $\cos. \theta = -\frac{1}{2}$.

(4) When $\sec. \theta = -1$.

(5) When $\sec.^2 \theta = 2$.

(6) When $\tan.^2 \theta = 3$.

CHAPTER X.

ON THE TRIGONOMETRICAL RATIOS OF THE SUM OR DIFFERENCE OF ANGLES.

59. THE object of the present chapter is to establish formulæ for expressing the ratios of angles made up of the sum or difference of other angles in terms of functions of these angles themselves.

60. The formulæ first to be established are as follows:—

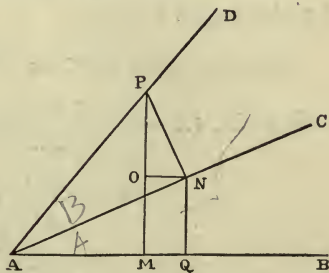
$$\text{Sin. } (A + B) = \sin. A \cos. B + \cos. A \sin. B$$
$$\text{Cos. } (A + B) = \cos. A \cos. B - \sin. A \sin. B$$
$$\text{Sin. } (A - B) = \sin. A \cos. B - \cos. A \sin. B$$
$$\text{Cos. } (A - B) = \cos. A \cos. B + \sin. A \sin. B$$

61. *To show that*

$$\text{Sin. } (A + B) = \sin. A \cos. B + \cos. A \sin. B$$
$$\text{Cos. } (A + B) = \cos. A \cos. B - \sin. A \sin. B$$

Let the angle BAC be represented by A, and the angle CAD by B.

Then the angle BAD will be represented by A + B.

From P any point in the line AD, draw PM at right angles to AB, and PN at right angles to AC.

From N draw NO at right angles to PM, and NQ at right angles to AB.

Then the angle OPN = 90° – PNO = ONA = NAQ = A.

$$\text{Now sin. }(A + B) = \frac{PM}{AP} = \frac{OM + OP}{AP} = \frac{NQ + OP}{AP}$$

$$= \frac{NQ}{AP} + \frac{OP}{AP} = \frac{NQ}{AN} \cdot \frac{AN}{AP} + \frac{OP}{PN} \cdot \frac{PN}{AP}$$

$$= \text{sin. } A \,.\, \text{cos. } B + \text{cos. } A \text{ sin. } B;$$

$$\text{and cos. }(A + B) = \frac{AM}{AP} = \frac{AQ - QM}{AP} = \frac{AQ - ON}{AP}$$

$$= \frac{AQ}{AN} \cdot \frac{AN}{AP} - \frac{ON}{PN} \cdot \frac{PN}{AP}$$

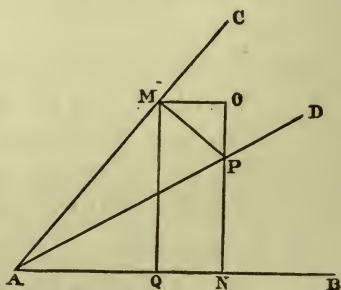
$$= \text{cos. } A \,.\, \text{cos. } B - \text{sin. } A \text{ sin. } B.$$

62. *To show that*

$$\text{Sin. }(A - B) = \text{sin. } A \text{ cos. } B - \text{cos. } A \text{ sin. } B$$
$$\text{Cos. }(A - B) = \text{cos. } A \text{ cos. } B + \text{sin. } A \text{ sin. } B.$$

Let the angle BAC be represented by A, and the angle CAD by B.

Then the angle BAD will represent A – B.

Take any point P in AD, and from P draw PM at right angles to AC, and PN at right angles to AB.

From M draw MO at right angles to NP produced, and MQ at right angles to AB.

Then the angle MPO = 90° – OMP = CMO = A.

$$\text{Now sin. }(A - B) = \frac{PN}{AP} = \frac{ON - OP}{AP} = \frac{MQ - OP}{AP}$$

$$= \frac{MQ}{AP} - \frac{OP}{AP} = \frac{MQ}{AM} \cdot \frac{AM}{AP} - \frac{OP}{PM} \cdot \frac{PM}{AP}$$

$$= \text{sin. } A \text{ cos. } B - \text{cos. } A \text{ sin. } B;$$

$$\text{and cos. } (A - B) = \frac{AN}{AP} = \frac{AQ + QN}{AP} = \frac{AQ + OM}{AP}$$

$$= \frac{AQ}{AP} + \frac{OM}{AP}$$

$$= \frac{AQ}{AM} \cdot \frac{AM}{AP} + \frac{OM}{MP} \cdot \frac{MP}{AP}$$

$$= \text{cos. } A \text{ cos. } B + \text{sin. } A \text{ . sin. } B.$$

In the construction of the figure of this and the preceding article, it may assist the student to notice that P, the point from which perpendiculars are let fall, lies in the line which bounds the compound angle A+B or A−B.

63. Expressions for tan. (A + B) and tan. (A − B), in terms of tan. A and tan. B, may be established independently by means of the figures given in arts. 61 and 62. It is, however, simpler to deduce them from the formulæ already established.

$$\text{Thus, tan. } (A + B) = \frac{\text{sin. } (A + B)}{\text{cos. } (A + B)}$$

$$= \frac{\text{sin. } A \text{ cos. } B + \text{cos. } A \text{ sin. } B}{\text{cos. } A \text{ cos. } B - \text{sin. } A \text{ sin. } B},$$

and this by dividing each term of numerator and denominator by cos. A cos. B becomes

$$= \frac{\dfrac{\text{sin. } A \text{ cos. } B}{\text{cos. } A \text{ cos. } B} + \dfrac{\text{cos. } A \text{ sin. } B}{\text{cos. } A \text{ cos. } B}}{\dfrac{\text{cos. } A \text{ cos. } B}{\text{cos. } A \text{ cos. } B} - \dfrac{\text{sin. } A \text{ sin. } B}{\text{cos. } A \text{ cos. } B}} = \frac{\text{tan. } A + \text{tan. } B}{1 - \text{tan. } A \text{ tan. } B}$$

$$\text{Again, tan. } (A - B) = \frac{\text{sin. } (A - B)}{\text{cos. } (A - B)}$$

$$= \frac{\text{sin. } A \text{ cos. } B - \text{cos. } A \text{ sin. } B}{\text{cos. } A. \text{ cos. } B + \text{sin. } A \text{ sin. } B}$$

$$= \frac{\dfrac{\text{sin. } A \text{ cos. } B}{\text{cos. } A \text{ cos. } B} - \dfrac{\text{cos. } A \text{ sin. } B}{\text{cos. } A \text{ cos. } B}}{\dfrac{\text{cos. } A \text{ cos. } B}{\text{cos. } A \text{ cos. } B} + \dfrac{\text{sin. } A \text{ sin. } B}{\text{cos. } A \text{ cos. } B}}$$

$$= \frac{\text{tan. } A - \text{tan. } B}{1 + \text{tan. } A \text{ tan. } B}.$$

In the same way may be deduced the following results :—

$$\cot. (A + B) = \frac{\cot. A \cot. B - 1}{\cot. A + \cot. B};$$

$$\cot. (A - B) = \frac{\cot. A \cot. B + 1}{\cot. B - \cot. A}.$$

64. The formulæ now established will enable us to find the values of the sines and cosines of 75° and 15°.

Thus, to find the value of sin. 75°.

$$\text{Sin.}\, 75° = \sin. (45° + 30°) = \sin. 45° \cos. 30° + \cos. 45° \sin. 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

Again, to find the value of sin. 15°, we have

$$\sin. 15° = \sin. (45° - 30°) = \sin. 45° \cos. 30° - \cos. 45° \sin. 30°.$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

65. The diagrams of arts. 61, 62 are constructed for the simplest cases; thus, in art. 61, A, B are together less than 90°, while in art. 62 A is less than 90°, and B less than A.

The formulæ obtained, however, are universally true, as may be shown in any particular case.

Thus, for example, let A, B each lie between 45° and 90°, so that $A + B$ lies between 90° and 180°.

Let $A' = 90° - A$.
$B' = 90° - B$.

Then $\sin. (A' + B') = \sin. (180° - \overline{A + B}) = \sin. (A + B)$.

Since A, B are each greater than 45°, it follows that $A' + B'$ is less than 90°.

Therefore, by making use of the diagram of art. 61, we have

$$\sin. (A' + B') = \sin. A' \cos. B' + \cos. A' \sin. B'$$
$$= \cos. A \sin. B + \sin. A \cos. B.$$

And from above, $\sin. (A' + B') = \sin. (A + B)$.

Therefore $\sin. (A + B) = \sin. A \cos. B + \cos. A \sin. B$.

And $\cos.(A'+B')=\cos.(180^\circ-\overline{A+B})= -\cos.(A+B)$.

But by art. 61

$$\cos.(A'+B')=\cos. A' \cos. B'-\sin. A' \sin. B'$$
$$=\sin. A \sin. B-\cos. A \cos. B.$$

Therefore

$$\cos.(A+B)= -\cos.(A'+B')=\cos. A \cos. B-\sin. A \sin. B.$$

In this particular instance the value of $A-B$ is clearly less than 90°, and the values of $\sin.(A-B)$, $\cos.(A-B)$ may therefore be deduced directly from the diagram of art. 62.

By processes similar to the above the expansions for $A+B$ and $A-B$ may be shown to be true for all values of A and B.

Examples.—X.

Show that the following relations are true:—

1. $\text{Sin.}(A+B)\cos. A-\cos.(A+B)\sin. A=\sin. B.$
2. $\text{Cos.}(A+B)\cos. A+\sin.(A+B)\sin. A=\cos. B.$
3. $\dfrac{\sin.(A+B)}{\cos. A \cos. B}=\tan. A+\tan. B.$
4. $\dfrac{\sin.(A+B)\sin.(A-B)}{\cos.^2 A \cos.^2 B}=\tan.^2 A-\tan.^2 B.$
5. $\text{Cos.}(A+B)\cos.(A-B)=\cos.^2 A-\sin.^2 B.$
6. $\text{Sin.}(A+B)\sin.(A-B)=\cos.^2 B-\cos.^2 A.$
7. $\dfrac{\tan. A \cot. B+1}{\cot. B-\tan. A}=\tan.(A+B).$
8. $\dfrac{\tan.(A+B)-\tan. A}{1+\tan.(A+B)\tan. A}=\tan. B.$
9. $\dfrac{\tan.(A-B)+\tan. B}{1-\tan.(A-B)\tan. B}=\tan. A.$
10. $\dfrac{\cot. A+\cot. B}{\cot. B-\cot. A}=\dfrac{\sin.(A+B)}{\sin.(A-B)}.$
11. $(\sin. A-\sin. B)^2+(\cos. A-\cos. B)^2=2\text{ vers.}(A-B).$
12. $\text{Cot. A}\cot. B\cos.(A+B)=\cos. A\cos. B(\cot. A\cot. B-1).$
13. $\text{Sec.}(A+B)=\dfrac{\sec. A \sec. B}{1-\tan. A \tan. B}.$

14. Cos. A − sin. A = $\sqrt{2}$ cos. (A + 45°).

15. $\dfrac{1 - \text{tan. } (45° - A)}{1 + \text{tan. } (45° - A)} = \text{tan. } A.$

16. Cos. 80° + 2 sin. 10° sin. 70° = $\frac{1}{2}$.

17. If sin. A = $\frac{8}{17}$, and cos. B = $\frac{5}{13}$, find the value of sin. (A + B).

18. If sin. A = $\frac{1}{\sqrt{10}}$, and cos. B = $\frac{3}{5}$, find the value of tan. (A + B).

19. Find the value of cos. 105°, of tan. 75°, and of tan. 15°.

20. If tan. A = $\frac{4}{5}$, and tan. B = $\frac{1}{9}$, show that A + B = 45°.

EXAMPLES. –XI.

Apply the formulæ established in this chapter to prove the following relations :—

1. Cos. (180° + A) = − cos. A.
2. Sin. (90° + A) = cos. A.
3. Cos. (360° − A) = cos. A.
4. Cos. (180° + A) = cos. (180° − A).
5. Cos. (270° + A) = sin. A.
6. Tan. (225° + A) = cot. (225° − A).

66. In the reduction of trigonometrical expressions, it is often an advantage when we meet with an expression involving the sum or difference of functions of angles to transform it into another containing only a product.

The values obtained for sin. (A + B), cos. (A + B), &c. in terms of sines and cosines of A and B may usefully be employed for this purpose.

Thus

Sin. A = sin. $\{\frac{1}{2}(A + B) + \frac{1}{2}(A - B)\}$ =
sin. $\frac{1}{2}$(A+B) cos. $\frac{1}{2}$(A−B) + cos. $\frac{1}{2}$(A+B) sin. $\frac{1}{2}$(A−B)

Sin. B = sin. $\{\frac{1}{2}(A + B) - \frac{1}{2}(A - B)\}$ =
sin. $\frac{1}{2}$(A+B) cos. $\frac{1}{2}$(A−B) − cos. $\frac{1}{2}$(A+B) sin. $\frac{1}{2}$(A−B)

$\text{Cos. } A = \cos. \{\frac{1}{2}(A + B) + \frac{1}{2}(A - B)\} =$
$\cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B) - \sin. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)$
$\text{Cos. } B = \cos. \{\frac{1}{2}(A + B) - \frac{1}{2}(A - B)\} =$
$\cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B) + \sin. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B).$

Adding and subtracting these four expressions, two by two, we arrive at the following results:—

$\text{Sin. } A + \sin. B = 2 \sin. \frac{1}{2}(A + B) \cos. \frac{1}{2}(A - B)$
$\text{Sin. } A - \sin. B = 2 \cos. \frac{1}{2}(A + B) \sin. \frac{1}{2}(A - B)$
$\text{Cos. } A + \cos. B = 2 \cos. \frac{1}{2}(A + B) \cos. \frac{1}{2}(A - B)$
$\text{Cos. } B - \cos. A = 2 \sin. \frac{1}{2}(A + B) \sin. \frac{1}{2}(A - B)$

67. These results are of great importance, and should be committed to memory. Expressed in words they may be stated as follows:—

(1) *The sum of the sines of two angles is equal to twice the sine of half their sum multiplied by the cosine of half their difference.*

(2) *The difference between the sines of two angles is equal to twice the cosine of half their sum multiplied by the sine of half their difference.*

(3) *The sum of the cosines of two angles is equal to twice the cosine of half their sum multiplied by the cosine of half their difference.*

(4) *The cosine of the lesser of two angles minus the cosine of the greater is equal to twice the sine of half their sum multiplied by the sine of half their difference.*

Example.—Express sin. 9A + sin. 3A in the form of a product.

Here half the sum is 6A, half the difference 3A.

Hence by rule (1) we have

$$\sin. 9A + \sin. 3A = 2 \sin. 6A \cos. 3A.$$

If the fundamental formulæ of art. 60 are added and subtracted, two by two, we shall obtain the expressions

$$\sin. (A + B) + \sin. (A - B) = 2 \sin. A \cos. B$$
$$\sin. (A + B) - \sin. (A - B) = 2 \cos. A \sin. B$$
$$\cos. (A + B) + \cos. (A - B) = 2 \cos. A \cos. B$$
$$\cos. (A - B) - \cos. (A + B) = 2 \sin. A \sin. B$$

The formulæ in this shape are convenient for converting an expression given as a product of sines and cosines into the form of a sum or difference.

Thus

$$2 \sin. 5A \sin. 2A = \cos. (5A - 2A) - \cos. (5A + 2A)$$
$$= \cos. 3A - \cos. 7A.$$

EXAMPLES.—XII.

Prove the following relations:—

1. $\text{Sin. } 8A + \sin. 2A = 2 \sin. 5A \cos. 3A.$
2. $\text{Cos. } A - \cos. 5A = 2 \sin. 3A \sin. 2A.$
3. $\text{Cos. } A + \cos. 4A = 2 \cos. \frac{5A}{2} \cos. \frac{3A}{2}.$
4. $\dfrac{\text{Sin. } A + \sin. 2A}{\cos. A - \cos. 2A} = \cot. \dfrac{A}{2}.$
5. $\dfrac{\text{Sin. } 3A + \sin. A}{\cos. 3A + \cos. A} = \tan. 2A.$
6. $\dfrac{\text{Sin. } A - \sin. B}{\cos. B - \cos. A} = \cot. \dfrac{A + B}{2}.$
7. $\text{Sin. } (30° + A) + \sin. (30° - A) = \cos. A$
8. $\text{Cos. } (30° - A) - \cos. (30° + A) = \sin. A.$
9. $\text{Sin. } (45° + A) - \sin. (45° - A) = \sqrt{2} \sin. A.$
10. $\text{Cos. } (45° + A) + \cos. (45° - A) = \sqrt{2} \cos. A.$
11. $\text{Cos. } A + \cos. (A + 2B) = 2 \cos. (A + B) \cos. B.$
12. $\text{Cos. } (A + B) \sin. B - \cos. (A + C) \sin. C$
$= \sin. (A + B) \cos. B - \sin. (A + C) \cos. C.$
13. $\text{Sin. } (A + B - C) + \sin. (A + C - B)$
$+ \sin. (B + C - A) - \sin. (A + B + C) = 4 \sin. A \sin. B \sin. C.$
14. $\text{Cos. } (A + B - C) + \cos. (A + C - B) + \cos. (B + C - A)$
$+ \cos. (A + B + C) = 4 \cos. A \cos. B \cos. C.$
15. $\text{Cosec. } 18° - \text{cosec. } 54° = 2.$
16. $\dfrac{\text{Cos. } 18° + \sin. 18°}{\cos. 18° - \sin. 18°} = \tan. 63°.$
17. $\text{Sin. } 70° - \sin. 10° = \cos. 40°.$
18. $\text{Sin. } 10° + \sin. 20° + \sin. 40° + \sin. 50° = \sin. 70°$
$+ \sin. 80°.$

19. Cos. 181° + cos. 127° + cos. 113° + cos. 61° + cos. 59° + cos. 7° = 0.

20. Cos. 4° 30′ + sin. 7° 30′ + sin. 37° 30′ − cos. 40° 30′ = 4 sin. 22° 30′ cos. 28° 30′ cos. 43° 30′.

EXAMPLES.—XIII.

Express as the sum or difference of two trigonometrical ratios the following expressions:—

1. 2 cos. A cos. B.
2. 2 sin. 4A cos. A.
3. 2 sin. 3A sin. 4A.
4. $2 \cos. \frac{9A}{2} \sin. \frac{5A}{2}$.
5. 2 cos. (A + B) sin. A.
6. $2 \cos. \frac{A+3B}{2} \cos. \frac{A}{2}$.
7. 2 sin. 50° cos. 20°.
8. 2 sin. 25° sin. 10°.
9. Show that sin. 4A cos. A − sin. A cos. 2A = sin. 3A cos. 2A.
10. Show that

$$\cos. \frac{5A}{2} \cos. \frac{3A}{2} - \sin. \frac{3A}{2} \sin. \frac{A}{2} = \cos. 3A \cos. A.$$

CHAPTER XI.

ON THE TRIGONOMETRICAL RATIOS FOR MULTIPLE AND SUB-MULTIPLE ANGLES.

68. If in the formula sin. (A + B) = sin. A cos. B + cos. A sin. B we make A equal to B, we obtain

$$\sin. 2A = \sin. (A + A) = \sin. A \cos. A + \cos. A \sin. A$$
$$= 2 \sin. A \cos. A.$$

Similarly $\cos. 2A = \cos. (A + A) = \cos. A \cos. A - \sin. A \sin. A.$

$$= \cos.^2 A - \sin.^2 A = 2 \cos.^2 A - 1,$$
$$\text{or } = 1 - 2 \sin.^2 A.$$

$$\text{Again, } \tan. 2A = \tan. (A + A) = \frac{\tan. A + \tan. A}{1 - \tan. A . \tan. A} = \frac{2 \tan. A}{1 - \tan^2 A}.$$

If for 2A we substitute A, and consequently for A write $\frac{A}{2}$, we shall have

$$\sin. A = 2 \sin. \frac{A}{2} \cos. \frac{A}{2};$$

$$\cos. A = \cos.^2 \frac{A}{2} - \sin.^2 \frac{A}{2} = 2 \cos.^2 \frac{A}{2} - 1, \text{ or } 1 - 2 \sin.^2 \frac{A}{2};$$

$$\tan. A = \frac{2 \tan. \frac{A}{2}}{1 - \tan.^2 \frac{A}{2}}.$$

EXAMPLES.—XIV.

Prove the following identities :—

1. $\text{Cos. } 2A = \frac{1 - \tan.^2 A}{1 + \tan.^2 A}$.
2. $\text{Tan. } 2A = \frac{2 \cot. A}{\cot.^2 A - 1}$.
3. $\frac{\text{Sec. } 2A - 1}{\sec. 2A + 1} = \tan.^2 A$.
4. $\left(\text{Sin. } \frac{A}{2} + \cos. \frac{A}{2}\right)^2 = 1 + \sin. A$.
5. $\text{Cos.}^2 \frac{A}{2} \left(1 + \tan. \frac{A}{2}\right)^2 = 1 + \sin. A$.
6. $\text{Cos.}^4 A - \sin.^4 A = \cos. 2A$.
7. $\frac{\text{Sec.}^2 A}{1 - \tan.^2 A} = \sec. 2A$.
8. $\frac{\text{Cosec.}^2 A}{\cot.^2 A - 1} = \sec. 2A$.
9. $\frac{1 - \cos. A}{1 + \cos. A} = \tan.^2 \frac{A}{2}$.
10. $\text{Tan.}^2\left(45° + \frac{A}{2}\right) = \frac{1 + \sin. A}{1 - \sin. A}$.
11. $\frac{\text{Cos. } A + \sin. A}{\cos. A - \sin. A} = \tan. 2A + \sec. 2A$.
12. $\frac{\text{Cos.}^3 A + \sin.^3 A}{\cos. A + \sin. A} = \frac{2 - \sin. 2A}{2}$.
13. $\frac{1 + \sin. A - \cos. A}{1 + \sin. A + \cos. A} = \tan. \frac{A}{2}$.

14. $2 \cos.^8 A - 2 \sin.^8 A = \cos. 2A (1 + \cos.^2 2A)$.

15. $\dfrac{\text{Sin. } A + \sin. 2A}{1 + \cos. A + \cos. 2A} = \tan. A$.

16. $\dfrac{1 + \sin. A}{1 + \cos. A} = \dfrac{1}{2}\left(1 + \tan. \dfrac{A}{2}\right)^2$.

17. $\dfrac{\text{Cos. } (A + 45^\circ)}{\cos. (A - 45^\circ)} = \sec. 2A - \tan. 2A$.

18. $\dfrac{1 + \tan. A \tan. \dfrac{A}{2}}{\tan. \dfrac{A}{2} + \cot. \dfrac{A}{2}} = \dfrac{1}{2} \tan. A$.

19. $\text{Tan.} \dfrac{180^\circ - 2A}{4} + \tan. \dfrac{180^\circ + 2A}{4} = 2 \sec. A$.

20. $\text{Cos.}^6 A - \sin.^6 A = \cos. 2A (\cos.^2 2A + \dfrac{3}{4} \sin.^2 2A)$.

69. In the same manner as in art. 68 it is easy to deduce the values of sin. 3A, cos. 3A, and tan. 3A in terms of sines, cosines, and tangents of A respectively.

Thus

$$\begin{aligned} \sin. 3A &= \sin. (2A + A) \\ &= \sin. 2A \cos. A + \cos. 2A \sin. A \\ &= 2 \sin. A \cos. A \cos. A + (1 - 2 \sin.^2 A) \sin. A \\ &= 2 \sin. A \cos.^2 A + \sin. A - 2 \sin.^3 A \\ &= 2 \sin. A - 2 \sin.^3 A + \sin. A - 2 \sin.^3 A \\ &= 3 \sin. A - 4 \sin.^3 A. \end{aligned}$$

70.

$$\begin{aligned} \text{Cos. } 3A &= \cos. (2A + A) \\ &= \cos. 2A. \cos. A - \sin. 2A \sin. A. \\ &= (2 \cos.^2 A - 1) \cos. A - (2 \sin. A \cos. A) \sin. A \\ &= 2 \cos.^3 A - \cos. A - 2 \sin.^2 A \cos. A \\ &= 2 \cos.^3 A - \cos. A - 2 \cos. A + 2 \cos.^3 A \\ &= 4 \cos.^3 A - 3 \cos. A. \end{aligned}$$

71.

$$\begin{aligned} \text{Tan. } 3A &= \tan. (2A + A) \\ &= \frac{\tan. 2A + \tan. A}{1 - \tan. 2A \tan. A} \\ &= \frac{\dfrac{2 \tan. A}{1 - \tan.^2 A} + \tan. A}{1 - \dfrac{2 \tan. A . \tan. A}{1 - \tan.^2 A}} \end{aligned}$$

$$=\frac{\dfrac{2\ \text{tan.}\ A+\text{tan.}\ A-\text{tan.}^3\ A}{1-\text{tan.}^2\ A}}{\dfrac{1-\text{tan.}^2\ A-2\ \text{tan.}^2\ A}{1-\text{tan.}^2\ A}}$$

$$=\frac{3\ \text{tan.}\ A-\text{tan.}^3\ A}{1-3\ \text{tan.}^2\ A}.$$

Examples.—XV.

Prove the identities

1. $\dfrac{\text{Sin. } 3A}{\text{sin. } A}=2\ \text{cos. } 2A+1.$

2. $\dfrac{\text{Cos. } 3A}{\text{cos. } A}=2\ \text{cos. } 2A-1.$

3. $\dfrac{3\ \text{sin. } A-\text{sin. } 3A}{\text{cos. } 3A+3\ \text{cos. } A}=\text{tan.}^3\ A.$

4. $\dfrac{\text{Sin. } 3A-\text{cos. } 3A}{\text{sin. } A+\text{cos. } A}=2\ \text{sin. } 2A-1.$

5. $\dfrac{1}{\text{tan. } 3A-\text{tan. } A}+\dfrac{1}{\text{cot. } A-\text{cot. } 3A}=\text{cot. } 2A.$

6. $4\ \text{sin. } \dfrac{A}{3}\ \text{sin. } \dfrac{180^\circ-A}{3}\ \text{sin. } \dfrac{180^\circ+A}{3}=\text{sin. } A.$

7. $\dfrac{\text{Tan. } (3A-45^\circ)}{\text{tan. } (A+45^\circ)}=\dfrac{2\ \text{sin. } 2A-1}{2\ \text{sin. } 2A+1}.$

8. $4\ \text{cos.}^3\ \theta\ \text{sin. } 3\theta+4\ \text{sin.}^3\ \theta\ \text{cos. } 3\theta=3\ \text{sin. } 4\theta.$

9. $\text{Cos. } 4A=8\ \text{cos.}^4\ A-8\ \text{cos.}^2\ A+1.$

10. $\text{Tan. } 4A=\dfrac{4\ \text{tan. } A-4\ \text{tan.}^3\ A}{1-6\ \text{tan.}^2\ A+\text{tan.}^4\ A}.$

11. $\text{Sin. } (A+2B)=\dfrac{\text{sin. } A}{\text{sec. } 2B}+\dfrac{\text{cos. } A}{\text{cosec. } 2B}.$

12. $\text{Tan.}^2\ A+\text{cot.}^2\ A=2+4\ \text{cot.}^2\ 2A.$

13. $\text{Cot. } A-\text{tan. } A-2\ \text{tan. } 2A-4\ \text{cot. } 4A=0.$

14. $\text{Tan. } A\ \text{tan. } 2A=\text{sec. } 2A-1.$

15. $\text{Tan. } A\ \text{tan. } (A-B)\ (\text{cot. } A+\text{tan. } B)=\text{tan. } A-\text{tan. } B.$

16. $\text{Sin.}^2\ B+\text{sin.}^2\ (A-B)+2\ \text{sin. } B\ \text{sin. } (A-B)\ \text{cos. } A=\text{sin.}^2\ A.$

17. $\frac{\text{Sin.}^2 2A-4 \sin.^2 A}{\sin.^2 2A+4 \sin.^2 A-4}=\tan.^4 A.$

18. $\text{Cos.}(A+B)+\cos. A = \cos.(A-B)+\cos. 3A + 4\sin.\left(A-\frac{B}{2}\right)\sin. A \cos.\left(A+\frac{B}{2}\right).$

19. $\text{Sin.} 2A+\sin. 4A+\sin. 6A=4\cos. A\cos. 2A\sin. 3A.$

20. $\text{Vers.}(A-B)\text{ vers.}\{180°-(A+B)\}=(\sin. A-\sin. B)^2.$

21. $\frac{\text{Cos. } nA-\cos. (n+2) A}{\sin. (n+2) A-\sin. nA}=\tan. (n+1) A.$

22. $2\,\frac{\cot. (n-2) A \cot. n A+1}{\cot. (n-2) A-\cot. nA}=\cot. A-\tan. A.$

23. $\text{Tan.}(30°-A)\tan.(30°+A)=\frac{2\cos. 2A-1}{2\cos. 2A+1}.$

24. $\text{Sin.}^2(30°+A)+\sin.^2(30°-A)=1-\frac{1}{2}\cos. 2A.$

25. $\text{Tan.}(45°-A)+\tan.(45°+A)=2\sec. 2A.$

26. $\text{Cos.}(A-45°)\cos.(B-45°)+\cos.^2\left(\frac{A+B}{2}+45°\right) =\cos.^2\frac{A-B}{2}.$

27. $\text{Sin.} 6A=4\sin. 2A\sin.(60°+2A)\sin.(60°-2A).$

28. $\text{Cot.} A+\cot.(60°+A)+\cot.(120°+A)=3\cot. 3A.$

29. $\text{Tan.} 50°+\cot. 50°=2\sec. 10°.$

30. $\text{Tan.} 20°+\tan. 20°\tan. 25°+\tan. 25°=1.$

31. $\text{Cos.} 20°\cos. 40°\cos. 80°=\frac{1}{8}.$

32. $\text{Cot.} 22° 30'-\tan. 22° 30'=2.$

33. $\text{Cos.} 40°+\cos. 50°+\cos. 60°+\cos. 150° =4\cos. 45°\cos. 50°\cos. 55°.$

34. $\text{Cos.}(A+B+C) =\cos. A\cos. B\cos. C(1-\tan. B\tan. C-\tan. C\tan. A-\tan. A\tan. B).$

35. $\text{Sin.} 2(A-B)+\sin. 2(A-C)+\sin. 2(C-B) =4\cos.(B-C)\cos.(C-A)\sin.(A-B).$

36. $\text{Sin.} A+\sin. B-\sin. C-\sin.(A+B-C) =4\sin.\frac{A-C}{2}\sin.\frac{B-C}{2}\sin.\frac{A+B}{2}$

72. We now proceed to obtain formulæ for expressing the sine, cosine, and tangent of half an angle in terms of functions of the angle.

To prove that $\sin.\frac{A}{2} = \pm\sqrt{\frac{1-\cos. A}{2}}$.

Since $\cos. A = 1 - 2\sin.^2\frac{A}{2}$ (art. 68),

$$\therefore \quad 2\sin.^2\frac{A}{2} = 1 - \cos. A;$$

$$\therefore \quad \sin.^2\frac{A}{2} = \frac{1-\cos. A}{2};$$

$$\therefore \quad \sin.\frac{A}{2} = \pm\sqrt{\frac{1-\cos. A}{2}}.$$

73. *To prove that* $\cos.\frac{A}{2} = \pm\sqrt{\frac{1+\cos. A}{2}}$.

Since $\cos. A = 2\cos.^2\frac{A}{2} - 1$ (art. 68),

$$\therefore \quad 2\cos.^2\frac{A}{2} = 1 + \cos. A;$$

$$\therefore \quad \cos.^2\frac{A}{2} = \frac{1+\cos. A}{2};$$

$$\therefore \quad \cos.\frac{A}{2} = \pm\sqrt{\frac{1+\cos. A}{2}}.$$

From the ambiguity of sign it is clear that each value of cos. A (nothing else being known of the angle A) gives two values of $\sin.\frac{A}{2}$ and two values of $\cos.\frac{A}{2}$. If besides cos. A we have given the magnitude of A, so that we can tell in which quadrant the revolving line will fall, after describing the angle $\frac{A}{2}$, we are in a position to affix the proper signs to the root symbol, both for $\sin.\frac{A}{2}$ and $\cos.\frac{A}{2}$.

Thus, let it be given that $\cos. A = \frac{1}{2}$.

$$\text{Then } \sin.\frac{A}{2} = \pm\sqrt{\frac{1-\cos. A}{2}} = \pm\frac{1}{2};$$

$$\cos.\frac{A}{2} = \pm\sqrt{\frac{1+\cos. A}{2}} = \pm\frac{\sqrt{3}}{2}.$$

Now the positive angles less than 360° which have the given cosine are 60° and 300°, the negative angles are −60° and −300°.

If the angle be 300°, since $\frac{A}{2} = 150°$, we must take the positive sign for sin. $\frac{A}{2}$, and the negative for cos. $\frac{A}{2}$, and so on.

It is sufficient, therefore, to know in which quadrant the angle $\frac{A}{2}$ lies; we can then affix the appropriate signs of sin. $\frac{A}{2}$, cos. $\frac{A}{2}$ accordingly.

74. *To prove that*

$$2 \cos. \frac{A}{2} = \pm \sqrt{1 + \sin. A} \pm \sqrt{1 - \sin. A};$$

$$2 \sin. \frac{A}{2} = \pm \sqrt{1 + \sin. A} \mp \sqrt{1 - \sin. A}.$$

Since $\cos.^2 \frac{A}{2} + \sin.^2 \frac{A}{2} = 1,$

and $2 \cos. \frac{A}{2} \sin. \frac{A}{2} = \sin. A,$

$$\therefore \cos.^2 \frac{A}{2} + 2 \cos. \frac{A}{2} \sin. \frac{A}{2} + \sin.^2 \frac{A}{2} = 1 + \sin. A,$$

and $\cos.^2 \frac{A}{2} - 2 \cos. \frac{A}{2} \sin. \frac{A}{2} + \sin.^2 \frac{A}{2} = 1 - \sin. A.$

Hence, taking the square root of each side of the equations,

$$\cos. \frac{A}{2} + \sin. \frac{A}{2} = \pm \sqrt{1 + \sin. A},$$

and $\cos. \frac{A}{2} - \sin. \frac{A}{2} = \pm \sqrt{1 - \sin. A}.$

Therefore adding, we have

$$2 \cos. \frac{A}{2} = \pm \sqrt{1 + \sin. A} \pm \sqrt{1 - \sin. A},$$

and subtracting,

$$2 \sin. \frac{A}{2} = \pm \sqrt{1 + \sin. A} \mp \sqrt{1 - \sin. A}.$$

Hence cos. $\frac{A}{2}$ and sin. $\frac{A}{2}$ have each four values, corresponding to a given value of sin. A.

If the value of A be given we may remove the ambiguity by assigning the proper signs to the root symbols in the intermediate equations; thus

(1) $\cos. \frac{A}{2} + \sin. \frac{A}{2} = \pm \sqrt{1 + \sin. A}$.

(2) $\cos. \frac{A}{2} - \sin. \frac{A}{2} = \pm \sqrt{1 - \sin. A}$.

Let us suppose that $A = 210°$, so that $\sin. A = -\frac{1}{2}$.

Then, since $\frac{A}{2} = 105°$, we know that $\sin. \frac{A}{2}$ is positive, and $\cos. \frac{A}{2}$ negative.

Moreover $\sin. \frac{A}{2}$ is numerically greater than $\cos. \frac{A}{2}$.

Hence we must take the positive sign in equation (1), the negative in equation (2).

Thus

$$\cos. 105° + \sin. 105° = \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}};$$

$$\cos. 105° - \sin. 105° = -\sqrt{1 + \frac{1}{2}} = -\sqrt{\frac{3}{2}}.$$

Hence

$$\cos. 105° = \frac{1}{2\sqrt{2}}(1 - \sqrt{3});$$

$$\sin. 105° = \frac{1}{2\sqrt{2}}(1 + \sqrt{3}).$$

Here, also, it is not necessary to know the actual value of A, but only the limits between which A lies, the essential points being, in the first place, the signs belonging to $\cos. \frac{A}{2}$ and $\sin. \frac{A}{2}$, and in the second, the relative magnitude of $\cos. \frac{A}{2}$ and $\sin. \frac{A}{2}$.

75. Thus let it be given that A lies between 90° and − 90°.

Then $\frac{A}{2}$ lies between 45° and − 45°.

Thus $\cos. \frac{A}{2}$ is positive, and greater than $\sin. \frac{A}{2}$.

Therefore $\cos. \frac{A}{2} + \sin. \frac{A}{2} = \sqrt{1 + \sin. A}$;

$$\cos. \frac{A}{2} - \sin. \frac{A}{2} = \sqrt{1 - \sin. A};$$

Hence $2 \cos. \frac{A}{2} = \sqrt{1 + \sin. A} + \sqrt{1 - \sin. A}$;

$$2 \sin. \frac{A}{2} = \sqrt{1 + \sin. A} - \sqrt{1 - \sin. A}.$$

76. *To find the sine and cosine of an angle of* 18°.

Let A denote the angle which contains 18°; then 2A contains 36°, and 3A contains 54°.

Therefore $2A + 3A = 90°$.

Hence $\sin. 2A = \cos. 3A$.

Therefore $2 \sin. A \cos. A = 4 \cos.^3 A - 3 \cos. A$.

Dividing both sides by cos. A,

$$2 \sin. A = 4 \cos.^2 A - 3 = 4 - 4 \sin.^2 A - 3;$$

therefore $4 \sin.^2 A + 2 \sin. A - 1 = 0$.

Solving this as a quadratic equation, in which sin. A is the unknown quantity, we obtain

$$\sin. A = \frac{-1 \pm \sqrt{5}}{4}.$$

Since sin. 18° must be a positive quantity, the positive sign only is available for $\sqrt{5}$;

$$\text{therefore } \sin. A = \frac{\sqrt{5} - 1}{4};$$

$$\cos. 18° = \sqrt{1 - \sin.^2 18°} = \sqrt{1 - \frac{6 - 2\sqrt{5}}{16}} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

77. *To find the sine and cosine of an angle of* 36°.

Since $\cos. 36° = 1 - 2 \sin.^2 18°$;

therefore $\cos. 36° = 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2 = 1 - \frac{6 - 2\sqrt{5}}{8}$;

$$\therefore \cos. 36° = \frac{2 + 2\sqrt{5}}{8} = \frac{\sqrt{5} + 1}{4};$$

$$\text{and } \sin. 36° = \sqrt{1 - \cos.^2 36°} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

EXAMPLES.—XVI.

1. State the signs of

$$\left(\cos. \frac{A}{2} + \sin. \frac{A}{2}\right) \text{ and } \left(\cos. \frac{A}{2} - \sin. \frac{A}{2}\right)$$

when A has the following values :—

(1) 50°; (2) 160°; (3) 220°;
(4) 320°; (5) − 24°; (6) − 460°.

2. If A lies between 180° and 270°, show that

$$2 \sin. \frac{A}{2} = \sqrt{1 + \sin. A} + \sqrt{1 - \sin. A}.$$

3. If A lies between 450° and 630°, prove that

$$2 \cos. \frac{A}{2} = \sqrt{1 - \sin. A} - \sqrt{1 + \sin. A}.$$

4. Find the limits between which A must lie when

$$2 \cos. \frac{A}{2} = - \sqrt{1 + \sin. A} - \sqrt{1 - \sin. A}.$$

5. Find the values of

(1) sin. 72°; (2) cos. 54°; (3) sin. 9°; (4) cos. 9°; (5) cos. 81°.

CHAPTER XII.

ON THE SOLUTION OF TRIGONOMETRICAL EQUATIONS.

78. A TRIGONOMETRICAL EQUATION is a relation other than an identity between one or more functions of one or more angles. In solving the equation we obtain the value of one or other of the functions, whence we may determine, exactly or approximately, the magnitude of the angle with respect to which the given conditions are true.

As an example let us consider the equation

$\text{Cos.}^2\,\theta + 2\,\text{sin.}\,\theta = \frac{7}{4}.$

Our first step will be to express $\cos.^2\theta$ in terms of $\sin.\,\theta$. The equation may then be treated as an ordinary algebraical equation, in which $\sin.\,\theta$ takes the place of x.

Thus $1 - \sin.^2\,\theta + 2\,\sin.\,\theta = \frac{7}{4}$;

$$\therefore\ \sin.^2\,\theta - 2\,\sin.\,\theta = -\frac{3}{4};$$

$$\therefore\ \sin.^2\,\theta - 2\,\sin.\,\theta + 1 = \frac{1}{4};$$

$$\therefore\ \sin.\,\theta - 1 = \pm\frac{1}{2};$$

$$\text{and } \sin.\,\theta = \frac{3}{2} \text{ or } \frac{1}{2}.$$

The greater of these two values must be rejected, since the sine can never exceed unity.

The remaining value is the sine of 30°. Therefore one value of θ which satisfies the equation is $\frac{\pi}{6}$.

But by art. 56 all angles furnished by the formula $n\pi + (-1)^n\theta$ will have the same sine as θ.

Hence any one of these angles will satisfy the equation.

In the present instance $\theta = \frac{\pi}{6}$; $n\pi + (-1)^n\frac{\pi}{6}$ is therefore the general solution of the equation proposed.

79. As a second example take the equation

$$\text{Sin.}\,3\theta + \sin.\,2\theta + \sin.\,\theta = 0.$$

Since $\sin.\,3\theta + \sin.\,\theta = 2\,\sin.\,2\theta\,\cos.\,\theta$,

therefore $2\,\sin.\,2\theta\,\cos.\,\theta + \sin.\,2\theta = 0$,

or $\sin.\,2\theta\,(2\,\cos.\,\theta + 1) = 0$.

Thus the equation is satisfied if either

$$\sin.\,2\theta = 0, \text{ or } 2\,\cos.\,\theta + 1 = 0.$$

If $\sin.\,2\theta = 0$, we have $2\theta = 0$, and the general value is given by $2\theta = n\pi$.

If $2\,\cos.\,\theta + 1 = 0$, $\cos.\,\theta = -\frac{1}{2}$, and $\theta = \frac{2\pi}{3}$.

The general value is therefore, by art. 57,

$$\theta=2n\pi\pm\frac{2\pi}{3}.$$

80. It will sometimes be convenient to proceed as in the following example.

Thus, in the equation

$$\sqrt{3}\ \text{sin.}\ \theta-\text{cos.}\ \theta=\sqrt{2},$$

dividing both sides of the equation by 2 we obtain

$$\frac{\sqrt{3}}{2}\ \text{sin.}\ \theta-\frac{1}{2}\ \text{cos.}\ \theta=\frac{1}{\sqrt{2}};$$

therefore $\text{sin.}\ \theta\ \text{cos.}\ \frac{\pi}{6}-\text{cos.}\ \theta\ \text{sin.}\ \frac{\pi}{6}=\frac{1}{\sqrt{2}}$,

or $\text{sin.}\left(\theta-\frac{\pi}{6}\right)=\text{sin.}\ \frac{\pi}{4}$.

Hence $\theta-\frac{\pi}{6}=n\pi+(-1)^n\frac{\pi}{4}$.

EXAMPLES.—XVII.

Find values of θ not greater than two right angles which will satisfy the following equations :—

1. Sin. $\theta=$cosec. θ.
2. 4 cos. $\theta=3$ sec. θ.
3. Sin. θ. + cos. $\theta=0$.
4. 2 tan. $\theta=\text{sec.}^2\ \theta$.
5. Sin. $\theta+$cos. θ cot. $\theta=2$.
6. Cot. $\theta=$tan. $\frac{\theta}{2}$.
7. Cosec. $\theta-$sec. θ tan. $\theta=0$.
8. $\text{Cot.}^2\ \theta=\sqrt{2}$ cosec. $\theta-1$.
9. Sec. θ cosec. $\theta=4$ tan. θ.
10. $1+$cos. $\theta=\frac{3}{4}$ sec. θ.
11. Cot. $\theta=2\sqrt{3}$ sin. θ.
12. $\text{Cot.}^2\ \theta+4\ \text{cos.}^2\ \theta=3$.
13. 3 cot. $\theta-$sec. θ cosec. $\theta=1$.
14. $4\sqrt{3}$ cot. $\theta=7$ cosec. $\theta-4$ sin. θ.

15. Sin. θ (cos. $40° -$ cos. $20°$) $=$ cos. θ (sin. $40° +$ sin. $20°$).
16. Cos. $4\theta -$ cos. $2\theta = -1$.
17. Cos. $8\theta + 2 = 3$ cos. 4θ.

Examples.—XVIII.

Find all the values of θ which satisfy the equations—

1. Sin. $\theta = 1$.
2. Cot. $\theta = 1$.
3. Sec. $\theta = \sqrt{2}$.
4. Tan.$^2\, \theta = \frac{1}{3}$.
5. Cos. θ + sin. $\theta = \frac{1}{\sqrt{2}}$.
6. $\sqrt{2}$ sin. $\theta + \sqrt{2}$ cos. $\theta = \sqrt{3}$.
7. Tan. $\left(\frac{\pi}{4} + \theta\right)$ cot. $\left(\frac{\pi}{4} - \theta\right) = 3$.
8. 2 sec.$^2\, \theta + (2\sqrt{2} + 1)$ sec. $\theta + \sqrt{2} = 0$.
9. Cos. θ + cos. $\frac{1}{2}\theta$ = cos. $\frac{3}{4}\theta$.
10. Cos. θ + cos. 2θ + cos. $3\theta = 0$.

CHAPTER XIII.

ON THE INVERSE NOTATION.

81. In the equation sin. $\theta = a$, we state that the sine of a certain angle θ is a.

This equation is sometimes written in another form, viz. $\theta = \sin.^{-1} a$.

Thus $\sin.^{-1} a$ is an angle, and a mode of expressing the particular angle whose sine is a. For instance, sin. $30° = \frac{1}{2}$; therefore $30° = \sin.^{-1} \frac{1}{2}$.

82. The following example will illustrate the method of using this notation:—

Show that $\cos.^{-1}\frac{4}{5}+\cos.^{-1}\frac{3}{5}=90°$.

Let the angle $\cos.^{-1}\frac{4}{5}$ be denoted by θ, and the angle $\cos.^{-1}\frac{3}{5}$ by ϕ.

We have to show that $\theta+\phi=90°$.

That this may be the case sin. ϕ must be equal to cos. θ.

Now $\sin.\ \phi=\sqrt{1-\cos.^2\phi}=\sqrt{1-\frac{9}{25}}=\frac{4}{5}=\cos.\ \theta$.

Therefore θ is complementary to ϕ.

It must be remembered that in statements like the above the value of the angle indicated by $\cos.^{-1}\frac{3}{5}$ is its least positive value.

83. Again, let it be required to show that

$\text{Cos}.^{-1}a=2\sin.^{-1}\sqrt{\frac{1-a}{2}}$.

As before, let $\cos.^{-1}a=\theta$, $\sin.^{-1}\sqrt{\frac{1-a}{2}}=\phi$.

We have to show that $\theta=2\phi$.

If this be the case cos. θ will be equal to cos. 2ϕ.

Now $\cos.\ 2\phi=1-2\ \sin.^2\phi=1-2\cdot\frac{1-a}{2}=1-1+a=a$ $=\cos.\ \theta$.

Examples.—XIX.

Prove the following relations :—

1. $\text{Tan}.^{-1}\frac{4}{3}=2\tan.^{-1}\frac{1}{2}$.
2. $\text{Tan}.^{-1}2+\tan.^{-1}3=\frac{3\pi}{4}$.
3. $\text{Tan}.^{-1}\frac{1}{3}-\tan.^{-1}\frac{1}{4}=\tan.^{-1}\frac{1}{13}$.
4. $\text{Tan}.^{-1}\frac{1}{3}=\tan.^{-1}\frac{1}{2}+\tan.^{-1}\frac{1}{4}-\tan.^{-1}\frac{11}{27}$.
5. $\text{Sin}.^{-1}\frac{23}{27}=3\sin.^{-1}\frac{1}{3}$.

6. $\text{Tan.}^{-1}\frac{1}{7} + 4\tan.^{-1}\frac{1}{3} + 2\tan.^{-1}\frac{1}{2} = \frac{3\pi}{4}.$

7. $\text{Tan.}\left((\tan.^{-1}\frac{1}{5} + \tan.^{-1}\frac{2}{3}\right) = 1.$

8. $\text{Sin.}^{-1}\frac{4}{5} + 2\tan.^{-1}\frac{1}{3} = \frac{\pi}{2}.$

9. $\text{Cot.}(\text{cosec.}^{-1}a) = \sqrt{a^2 - 1}.$

10. $a\cos.\left(\sin.^{-1}\frac{b}{a}\right) = \sqrt{a^2 - b^2}.$

11. $\text{Sin. cot.}^{-1}a = \tan.\ \cos.^{-1}\left(\frac{a^2+1}{a^2+2}\right)^{\frac{1}{2}}.$

12. $\text{Tan.}^{-1}(2+\sqrt{3}) - \tan.^{-1}(2-\sqrt{3}) = \sec.^{-1}2.$

13. $\text{Cot.}^{-1}\sqrt{3} - \cot.^{-1}(\sqrt{3}+2) = \frac{\pi}{12}.$

14. $\{\text{Tan.}(\sin.^{-1}a) + \cot.(\cos.^{-1}a)\}^2 = 2a\tan.(2\tan.^{-1}a).$

Find the value of x in the following equations:—

15. $\text{Tan.}^{-1}\sqrt{x+1} = 2\tan.^{-1}\frac{1}{\sqrt{x+1}}.$

16. $\text{Sin.}^{-1}\frac{1}{x} = \frac{1}{3}\sin.^{-1}\frac{2}{x}.$

17. $\text{Cot.}^{-1}x + \cot.^{-1}(x-1) = \frac{\pi}{4}.$

18. $\text{Tan.}^{-1}\frac{1}{x-1} - \tan.^{-1}\frac{1}{x+1} = \tan.^{-1}a.$

19. $\text{Cos.}^{-1}x + \cos.^{-1}(1-x) = \cos.^{-1}(-x).$

20. $\text{Tan.}^{-1}x + \tan.^{-1}2x - \tan.^{-1}\frac{3\sqrt{3}}{5} = 0.$

CHAPTER XIV.

ON LOGARITHMS.

84. The logarithm of a number to a given base is the index of the power to which the base must be raised that it may produce the number.

Thus, if $m = a^x$, x is said to be the logarithm of m to the base a.

For example, $64 = 4^3$; then 3 is the logarithm of 64 to the base 4.

85. The logarithm of a number m to the base a is written thus:— $$x = \log._a m.$$

Hence, since $m = a^x$, $m = a^{\log._a m}$.

86. It follows from the definition that to construct a table of logarithms of a series of numbers, 1, 2, 3, . . . to a given base, as, for example, 10, we have to solve a series of equations: $10^x = 1$, $10^x = 2$, $10^x = 3$, . . .

These equations can in general be solved only approximately. Thus, for example, we cannot find a value of x which will make $10^x = 3$; we may, however, find such a value as will make 10^x differ from 3 by as small a quantity as we please.

87. We shall now establish some of the properties which render the use of logarithms advantageous in practical calculations.

88. *The logarithm of unity is 0, whatever the base may be.*

For we know by algebra that $a^0 = 1$, whatever may be the value of a.

89. *The logarithm of the base itself is unity.*

For $a^x = a$ when $x = 1$.

90. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Let $a^x = m$, $a^y = n$.

Then $x = \log._a m$, $y = \log._a n$.

And $a^x a^y = a^{x+y} = mn$.

Thus $\log._a mn = x + y = \log._a m + \log._a n$.

91. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

Let $a^x = m$, $a^y = n$.

Then $x = \log._a m$, $y = \log._a n$.

Therefore $\frac{a^x}{a^y} = a^{x-y} = \frac{m}{n}$.

Thus $\log._a \frac{m}{n} = x - y = \log._a m - \log._a n$.

92. *The logarithm of any power, integral or fractional, of a number is equal to the product of the logarithm of the number by the index of the power.*

For let $m = a^x$, so that $x = \log._a m$;
Therefore $m^r = (a^x)^r = a^{rx}$;
Thus $\log._a m^r = rx = r \log._a m$.

93. The following results flow from these important theorems:—

By the use of logarithms

Multiplication	is changed	into	Addition.
Division	,,	,,	Subtraction.
Involution	,,	,,	Multiplication.
Evolution	,,	,,	Division.

In practical calculations the only base that is used is 10, logarithms to the base 10 being known as *common* logarithms. We shall proceed to point out certain advantages which result from the employment of 10 as a base.

94. The logarithms of numbers in general consist of two parts, an integral part and a decimal part. The integral part is known as the *characteristic*, the decimal part as the *mantissa*.

95. *In the common system of logarithms, if the logarithm of any number be known, we can determine the logarithm of the product or quotient of that number by any power of* 10.

For $\log._{10} (10^n \times N) = \log._{10} N + \log._{10} 10^n = \log._{10} N + n$,

$$\log._{10} \frac{N}{10^n} = \log._{10} N - \log._{10} 10^n = \log._{10} N - n.$$

So that if we know the logarithm of a given number we may determine that of any number which has the same sequence of figures, and differs only in the position of the decimal point.

96. Thus, having given that $\log._{10} 5{\cdot}2502$ is $\cdot720176$,

let it be required to find the logarithm of $\log._{10} 5250 \cdot 2$ and $\log._{10} \cdot 0052502$.

$\text{Log.}_{10}\ 5250\cdot 2 = \log.\ (5\cdot 2502 \times 1000) = \log.\ 5\cdot 2502 + 3 = 3\cdot 720176.$

$$\text{Log.}_{10}\ \cdot 0052502 = \log.\ \frac{5\cdot 2502}{1000} = \log.\ 5\cdot 2502 - 3$$

$= -3 + \cdot 720176$, or, as it is generally written, $\bar{3}\cdot 720176$.

97. The form in which the logarithm of $\cdot 0052502$ is given should be very carefully noticed. This logarithm is really a negative quantity, and is equal to $-2\cdot 279824$.

In order, however, that all numbers expressed by the same series of digits may have the same *mantissa* in their logarithms, it is usual to agree that the *mantissa* shall in all cases remain positive.

This result is attained by simply adding unity to the *mantissa*, and subtracting it from the *characteristic*, the value of the whole logarithm being unaltered.

Thus in the present instance the *mantissa* becomes $-\cdot 279824 + 1$, or $+\cdot 720176$.

The *characteristic* becomes $-2-1$, that is -3; and, as we have said, the whole logarithm is written $\bar{3}\cdot 720176$.

98. *In the common system of logarithms the characteristic of the logarithm of any number may be determined by inspection.*

For suppose the number to be greater than unity, and to lie between 10^n and 10^{n+1}.

Its logarithm must be greater than n, and less than $n + 1$.

Therefore the *characteristic*, or integral part of the logarithm, is n. Thus in the example given above the *characteristic* of $\log._{10} 5250\cdot 2$ is 3, for $5250\cdot 2$ lies between 1000, or 10^3, and 10000, or 10^4.

Next suppose the number to be less than unity, and to lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, that is, between 10^{-n} and $10^{-(n+1)}$.

The integral part of the logarithm would be $-n$ if both *characteristic* and *mantissa* were to retain the negative sign.

But since, to avoid the inconvenience arising from a negative

mantissa, unity has to be subtracted from the *characteristic*, the latter will become $-(n+1)$ instead of $-n$.

Thus, since ·0052502 lies between ·01 and ·001, or between $(10)^{-2}$ and $(10)^{-3}$, we know that -3 must in this case be taken for the *characteristic*.

99. The rules for determining at once the characteristic of the logarithm of a number may be stated as follows:

(1) *If the quantity be greater than unity the characteristic is positive, and is one less than the number of figures which form the integral part of the number.*

(2) *If the quantity be less than unity the characteristic is negative, and, when the quantity is expressed as a decimal fraction, is one more than the number of cyphers between the decimal point and the first significant figure to the right of the decimal point.*

100. As has been said, tables of logarithms for practical use are calculated to the base 10. The actual processes by which the numerical values of logarithms are determined do not fall within the scope of this treatise; it may, however, be stated that the logarithms which are first calculated have for their base an incommensurable quantity, known by the symbol e, which is the sum of the series—

$$2 + \frac{1}{2} + \frac{1}{2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \ldots \textit{ ad inf.},$$

and this series may be taken as approximately equal to 2·7182818.

Such logarithms are known as *Napierian* logarithms, from Napier, the inventor of logarithms; and sometimes *natural* logarithms, as being those which are first determined.

We shall proceed to show how, having given the logarithm of a number to the base e, we may deduce the logarithm of the number to the base 10.

101. *Having given the logarithm of a number to a particular base, to find the logarithm of the number to a different base.*

Let a be the given base, b the other base, m the number.

Let x be the logarithm to base a, y the logarithm to the base b.

Then $m = a^x = b^y$.

Therefore $a^{\frac{x}{y}} = b$,

and $\frac{x}{y} = \log._a b$.

Therefore $x = y \log._a b$, and $y = \frac{x}{\log._a b}$;

Therefore $\log._b m = \frac{1}{\log._a b} \log._a m$.

Thus, if we have given the logarithms of a series of numbers calculated to a given base a, we may find the logarithms of these numbers to the base b by multiplying each of the given logarithms by the constant factor $\frac{1}{\log._a b}$.

102. In the particular case before us the number represented by e corresponds to a, while 10 may be written for b. Thus

$$\log._{10} m = \frac{1}{\log._e 10} \log._e m.$$

If, therefore, we assume that the logarithms of numbers may be calculated to the base e, it is clear that the logarithms of the same numbers to the base 10 may be deduced from them.

The value of the constant factor $\frac{1}{\log._e 10}$ is expressed approximately by the figures ·43429448: it is known as the *modulus* of the common system.

103. The following easy examples will illustrate the principles explained in this chapter. When no other base is indicated the number 10 is to be understood as the base.

Example 1.—Given $\log. 2 = \cdot 301030$.

$\log. 3 = \cdot 477121$,

find $\log. 12$ and $\log. \frac{\sqrt{2}}{\sqrt[3]{3}}$.

$\text{Log.}\ 12 = \log. (4 \times 3) = \log. 4 + \log. 3 = 2 \log. 2 + \log. 3$
$= \cdot 602060 + \cdot 477121 = 1 \cdot 079181$.

$$\text{Again, } \log. \frac{\sqrt{2}}{\sqrt[3]{3}} = \log. \sqrt{2} - \log. \sqrt[3]{3}$$
$$= \frac{1}{2} \log. 2 - \frac{1}{3} \log. 3$$
$$= \frac{1}{2} (\cdot 301030) - \frac{1}{3} (\cdot 477121)$$
$$= \cdot 150515 - \cdot 159040$$
$$= - \cdot 008525 = \bar{1} \cdot 991475.$$

Example 2.—Given $\log._{10} 7 = a$, find $\log._{7} 490$.

Let $\log._{7} 490 = x$.
Then $7^x = 490$;
Therefore $x \log._{10} 7 = \log._{10} 490$
$$= \log._{10} (49 \times 10) = \log._{10} 49 + \log._{10} 10$$
$$= 2 \log._{10} 7 + 1.$$
Hence $(x - 2) \log._{10} 7 = 1$;

therefore $x - 2 = \dfrac{1}{\log._{10} 7}$,

and $x = 2 + \dfrac{1}{\log._{10} 7} = \dfrac{2a + 1}{a}$.

EXAMPLES.—XX.

1. Given $\log. 2 = \cdot 301030$, and $\log. 3 = \cdot 477121$, find the logarithms of 30,000, ·0002, 2·25, and $\cdot 0\dot{4}$.

2. Given $\log._{10} 7 = \cdot 845098$, and $\log._{10} 6 \cdot 3 = \cdot 799341$, find $\log._{10} \dfrac{27}{49}$.

3. Given $\log. 3 = \cdot 477121$, find $\log. \{(2 \cdot 7)^3 \times (\cdot 81)^{\frac{4}{5}} \div (90)^{\frac{5}{4}}\}$.

4. Given $\log. 24 = 1 \cdot 380211$, and $\log. 36 = 1 \cdot 556303$, find the values of $\log._{10} 54$, and $\log._{9} 8$.

5. Find the logarithms of

(*a*) ·00001 to base 10.

(*b*) $\dfrac{1}{\sqrt{2}}$ to base 8.

(*c*) $81 \sqrt[3]{3}$ to base $3^{\frac{3}{2}}$.

6. Given $\log._{10} 2 = \cdot 301030$, find $\log._{10} \cdot 05$, $\log._{100} 2$, $\log._{2} 100$.

7. Having given log. $28 = 1{\cdot}447158$, and log. $35 = 1{\cdot}544068$, find $\log._{{\cdot}056} 7$ to 3 places of decimals.

8. Given log. $2 = {\cdot}301030$, and log. $3 = {\cdot}477121$, find x from the equation $3^x = 32$.

9. Given log. $2{\cdot}7 = {\cdot}431364$, and log. $5{\cdot}172818 = {\cdot}713727$, find the value of $3^{-\frac{3}{5}}$.

10. Find the logarithm of 9 to base $3\sqrt{3}$, and of 125 to base $\sqrt{5}\ \sqrt[3]{5}$.

11. If $\dfrac{4}{3}$ log. $(2^x) = 30{\cdot}103$, and log. $2 = {\cdot}301030$, find x.

12. Assuming that log. 250 and log. 256 differ by ${\cdot}0103$, find log. 2.

13. Given $\log._b a = c$, find $\log._a a^m b^m$.

14. Prove that $\log._a b . \log._b a = 1$, and that $\log._a b . \log._b c . \log._c a = 1$.

15. If the logarithms of the numbers a, b, and c be p, q, r respectively, prove that

$$a^{q-r}\, b^{r-p}\, c^{p-q} = 1.$$

16. If $a^2 + b^2 = 7ab$, prove that

$$\log. \frac{1}{3}(a + b) = \frac{1}{2}(\log. a + \log. b).$$

17. Having given log. $3 = {\cdot}477121$, and log. $7 = {\cdot}845098$, solve the equation

$$\left(\frac{7}{3}\right)^x = 100.$$

18. How many positive integers are there whose logarithms to base 3 have 5 for a characteristic?

19. Show how a system of logarithms having 2 for its base may be transformed into the system which has 8 for its base.

20. Prove that $\log._a x : \log._b x :: \log._c b : \log._c a$.

FORMULÆ OF REFERENCE (I.).

(1) $\sin.(90^\circ - A) = \cos. A$, $\cos.(90^\circ - A) = \sin. A$ (art. 41)

$\sin.(180^\circ - A) = \sin. A$, $\cos.(180^\circ - A) = -\cos. A$ (art. 42)

$\sin.(90^\circ + A) = \cos. A$, $\cos.(90^\circ + A) = -\sin. A$ (art. 43)

$\left.\begin{array}{l}\sin.(180^\circ + A) = -\sin. A., \cos.(180^\circ + A) = -\cos. A \\ \sin.(-A) = -\sin. A, \cos.(-A) = \cos. A\end{array}\right\}$ (art. 44)

(2) $\left.\begin{array}{l}\tan. A = \dfrac{\sin. A}{\cos. A}, \cot. A = \dfrac{\cos. A}{\sin. A}, \\ \sin.^2 A + \cos.^2 A = 1, \sec.^2 A = 1 + \tan.^2 A, \operatorname{cosec}.^2 A \\ \quad = 1 + \cot.^2 A\end{array}\right\}$ (art. 46)

(3) $\left.\begin{array}{l}\sin.(A+B) = \sin. A \cos. B + \cos. A \sin. B \\ \cos.(A+B) = \cos. A \cos. B - \sin. A \sin. B\end{array}\right\}$ . . (art. 61)

$\left.\begin{array}{l}\sin.(A-B) = \sin. A \cos. B - \cos. A \sin. B \\ \cos.(A-B) = \cos. A \cos. B + \sin. A \sin. B\end{array}\right\}$ . . (art. 62)

$\left.\begin{array}{l}\tan.(A+B) = \dfrac{\tan. A + \tan. B}{1 - \tan. A \tan. B} \\ \tan.(A-B) = \dfrac{\tan. A - \tan. B}{1 + \tan. A \tan. B}\end{array}\right\}$ . . . . . . (art. 63)

(4) $\left.\begin{array}{l}\sin. A + \sin. B = 2 \sin. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B) \\ \sin. A - \sin. B = 2 \cos. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B) \\ \cos. A + \cos. B = 2 \cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B) \\ \cos. B - \cos. A = 2 \sin. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)\end{array}\right\}$. (art. 66)

(5) $\left.\begin{array}{l}\sin. 2A = 2 \sin. A \cos. A \\ \cos. 2A = \cos.^2 A - \sin.^2 A \\ \tan. 2A = \dfrac{2 \tan. A}{1 - \tan.^2 A}\end{array}\right\}$ . . . . (art. 68)

(6) $\sin. 3A = 3 \sin. A - 4 \sin.^3 A$ (art. 69)

$\cos. 3A = 4 \cos.^3 A - 3 \cos. A$ (art. 70)

$\tan. 3A = \dfrac{3 \tan. A - \tan.^3 A}{1 - 3 \tan.^2 A}$ . . . . (art. 71)

(7)
$$\left.\begin{aligned}\sin.\frac{A}{2} &= \pm\sqrt{\frac{1-\cos. A}{2}}\\ \cos.\frac{A}{2} &= \pm\sqrt{\frac{1+\cos. A}{2}}\end{aligned}\right\} \quad \ldots \quad \text{(art. 73)}$$

$$\left.\begin{aligned}\cos.\frac{A}{2}+\sin.\frac{A}{2} &= \pm\sqrt{1+\sin. A}\\ \cos.\frac{A}{2}-\sin.\frac{A}{2} &= \pm\sqrt{1-\sin. A}\end{aligned}\right\} \quad \ldots \quad \text{(art. 74)}$$

(8) $\text{Log.}_a m\,n = \log._a m + \log._a n$ (art. 90)

$\log._a \frac{m}{n} = \log._a m - \log._a n$ (art. 91)

$\log._a m^r = r \log._a m$ (art. 92)

$\log._b m = \frac{1}{\log._a b} \log._a m$ (art. 101)

CHAPTER XV.

ON THE ARRANGEMENT OF LOGARITHMIC TABLES.

104. THE object of the present chapter is to give an account of the arrangement of the principal logarithmic and trigonometrical tables, and the method of using them. The tables which will be kept specially in view are those compiled by the late Dr. Inman, which are in general use in the Royal Navy. The various sets of tables, however, differ from one another chiefly in matters of detail, so that it is hoped that the explanations here given will enable the student to avail himself of any collection that may be at hand.

The logarithms in all cases are taken to the base 10.

105. The tables with which we are at present concerned are the following :—

(1.) The table of the logarithms of numbers from 1 to 10,000.
(2.) Tables of logarithms of the six principal trigonometrical

ratios, viz. the sine, cosecant, tangent, cotangent, secant, and cosine from 0° to 90°.

(3.) The table of the natural versines from 0° to 180°.

106. In the table which we have called (1) the mantissæ of the logarithms from 1 to 10,000 are given as far as six places of decimals. It should be noticed that in all approximate calculations it is usual to take for the last figure which is retained the figure which gives the nearest approach to the true value. Thus if we have the fraction ·7536, and it is desired to retain only three places of decimals, we should write ·754, and not ·753, the rule being that when a number of figures are struck off, the last remaining figure is to be increased by 1 if the first figure removed be not less than 5.

It is unnecessary to print the characteristic as well as the mantissa, because, as has been already pointed out, the characteristic may be always determined by inspection in the case of logarithms to the base 10.

107. *To find from the tables the logarithm of a given number.*

If the number be contained in the tables, we have only to take out the mantissa and prefix the characteristic.

Thus, to find the logarithm of ·527 and of ·00527. The table gives ·721811 as the mantissa of 527, and since the series of digits is the same in each case, the mantissa of the two logarithms will be the same. Thus we have

$$\log. 527 = 2\cdot721811. \qquad \log. \cdot00527 = \bar{3}\cdot721811.$$

If, however, the number be not exactly contained in the table, which, as we have said, goes only as far as 10,000, we must proceed as follows: Let us suppose that the logarithm required is that of 800076. The number in question lies between 800000 and 800100, both of which may be taken at once from the tables. Thus—

$$\begin{aligned} \log. 800100 &= 5\cdot903144 \\ \log. 800000 &= \underline{5\cdot903090} \\ \text{difference} &= \cdot000054 \end{aligned}$$

The required logarithm must of course lie between the two

which we have taken out, and if we may *assume* that the increase in the logarithm will be proportional to the increase in the number, we obtain the following proportion (where x is the increase required corresponding to an increase of 76 in the number):—

$$x : \cdot000054 :: 76 : 100;$$

$$\therefore x = \frac{76}{100} \cdot000054 = \cdot000041.$$

Then log. 800076 = 5·903090 + ·000041 = 5·903131.

Thus we have taken three numbers, 800000, 800076, and 800100, and have proceeded upon the assumption that the increments in the logarithms of the two higher numbers will be proportional to the increments of the numbers themselves.

This is a case of what is known as the *principle of proportional parts*, a principle which though not strictly true is sufficiently accurate for practical purposes in cases where the differences in the three numbers are small in comparison with the numbers themselves.

108. The process of finding the corresponding logarithm of a given number not contained in the tables is facilitated by the use of a small table known as the *table of proportional parts.* Thus in Inman's tables the logarithms of 8000 and the nine numbers next following are printed as follows:—

No.	Log.	Part.
8000	·903090	00
8001	·903144	05
8002	·903198	11
8003	·903253	16
8004	·903307	22
8005	·903361	27
8006	·903416	32
8007	·903470	38
8008	·903524	43
8009	·903578	49

To obtain the small table headed 'Part,' proceed as follows:

Take the difference between two consecutive logarithms of those given and divide by ten. Then multiply the resulting decimal by two, three, &c. up to nine, and retaining only six places of decimals write down the last three figures of these as in the table.

Thus in the preceding case

$$\begin{aligned} \text{log. } 8001 &= 3{\cdot}903144 \\ \text{log. } 8000 &= \underline{3{\cdot}903090} \\ \text{difference} &= {\cdot}000054 \end{aligned}$$

Therefore one-tenth of this difference is ·0000054, and multiplying by two, three, four, &c. we obtain ·0000108, ·0000162, ·0000216, &c. or, striking off the last figure in each case, ·000011, ·000016, ·000022, &c.

These are then arranged as in the table.

109. The use of this table may be illustrated by applying it in the example given above, viz. in finding the logarithm of 800076.

As we saw, when x is the excess of the logarithm required over that of 800000, we have

$$x = \frac{76}{100} \cdot 000054 = \left(\frac{7}{10} + \frac{6}{100}\right) \cdot 000054.$$

That is, we have to multiply ·000054 by $\frac{7}{10}$ and by $\frac{6}{100}$, and add the two products together.

The table of proportional parts gives us the value of the several tenth parts, and therefore also of the several hundredth parts of the decimal in question ; we may therefore proceed as follows :—

$$\begin{aligned} \text{log. } 800000 &= 5{\cdot}903090 \\ \text{add for } \tfrac{7}{10} &\quad\ {\cdot}000038 \\ \text{add for } \tfrac{6}{100} &\quad\ \underline{{\cdot}0000032} \\ &\quad\ 5{\cdot}9031312 \end{aligned}$$

Therefore, retaining only six places of decimals, we have

$$\text{log. } 800076 = 5{\cdot}903131.$$

110. The example given was the case of an integral number. If a decimal fraction or a mixed quantity, made up of a whole

number and a decimal fraction, were given, the process would be the same. We have only to disregard the decimal point, calculate the mantissa of the whole number so obtained as before, and then prefix the proper characteristic.

Thus, suppose the logarithm of 14·9037 to be required.

Rejecting the decimal point, we find, proceeding as before, the logarithm of 149037 to be 5·173294.

The logarithm of 14·9037 will only differ from this logarithm in its characteristic, which will be 1 instead of 5. The required logarithm, therefore, will be 1·173294.

111. *To find from the tables the number which corresponds to a given logarithm.*

If the mantissa be found in the table we have only to write down the corresponding number, and place the decimal point in the position indicated by the characteristic.

Thus, let the given logarithm be 4·155336. On reference to the table we find the mantissa given to be that of the logarithm of 143.

The required number is therefore 14300.

Next suppose that the mantissa is not contained exactly in the table, as, for instance, if the given logarithm be 5·348390. Here we must again have recourse to the '*principle of proportional parts.*' Turning to the tables, we find that the given logarithm lies between those of 223000 and 223100. Thus

$$\begin{array}{ll} \log.\ 223100 = 5{\cdot}348500 & \log.\ \text{given} \quad = 5{\cdot}348390 \\ \log.\ 223000 = \underline{5{\cdot}348305} & \log.\ 223000 = \underline{5{\cdot}348305} \\ \qquad\qquad\qquad\ \ {\cdot}000195 & \qquad\qquad\qquad\ \ {\cdot}000085 \end{array}$$

Thus, if x be the excess of the required number over 223000, we have

$$x : 100 :: {\cdot}000085 : {\cdot}000195;$$

$$\therefore x = \frac{85}{195} 100 = 44.$$

Thus the number required is 223044.

112. This process also may be somewhat abbreviated by recourse to the table of proportional parts.

Set down in full, the operation will be as follows :—

```
195) 8500 (43·589
     780
     ----
      700
      585
      ----
      1150
       975
      -----
       1750
       1560
       -----
        1900
        1755
        ----
```

The products 780, 585, 975, &c., are furnished approximately for us in the table of proportional parts. Thus, in the table which commences opposite to the number 2230, we find opposite to the figure 4, 078, that is, four-tenths of 195 = 78, or four times 195 = 780.

Using this table, therefore, we may proceed as follows:—

```
     8500
4    780
    -----
      700
3     580
     -----
      1200
6     1170
      ----
```

Thus we have 43·6 as the value for x, which is sufficiently near to the true value to be accurate enough for ordinary purposes. In cases where great exactness is required the actual division should, of course, be performed.

113. The table denoted by (2), next to be noticed, contains the logarithms of the six principal trigonometrical ratios, viz. the sine, cosecant, tangent, cotangent, secant, and cosine for all angles from 0° to 90°, calculated for each change of 15″ in the angle.

Since, however large an angle may be, by means of the expressions for sin. (180° − A), sin. (180° + A), &c. an angle may always be found in the first quadrant having for one of its functions the same *numerical* value as the required function of the given angle, it is unnecessary to take into account angles greater than 90°.

Thus cos. 125° = − cos. 55°, tan. 260° = tan. 80°, and so on.

Again, since sin. $(90° - A) = \cos. A$, tan. $(90° - A) = \cot. A$, cosec. $(90° - A) = \sec. A$, it is obvious that logarithms need not be calculated for angles higher than 45°. The logarithm of sin. 46°, for instance, is the same as that of cos. 44°. It will be seen on reference to the tables that the figures which denote the values of angles less than 45° are printed at the top and left-hand side of each page, and the figures for angles greater than 45° at the bottom and right-hand side; while each column of logarithms has a denomination both at the top and bottom, the column which is headed *sine* having *cosine* printed at the bottom, and so on.

114. The trigonometrical ratios being, in a majority of cases, less than unity, the logarithms of these ratios would have a negative characteristic. To avoid the frequent recurrence of the negative sign in the tables, it is customary to add 10 to the logarithms of the angles. Logarithms so increased are called *tabular logarithms*, and are denoted by the letter L.

$$\text{Thus L sin. A} = \text{log. sin. A} + 10.$$

115. *To find the tabular logarithm of the sine of a given angle.*

If the angle given be one of those contained in the tables the required sine may be at once taken out. If the angle be not given exactly we must have recourse to the principle of proportional parts.

Thus let it be required to find L sin. 40° 30′ 24″.

We find from the tables that

$$\begin{aligned} \text{L sin. } 40° \; 30' \; 30'' &= 9{\cdot}812618 \\ \text{L sin. } 40 \;\; 30 \;\; 15 \;\; &= \underline{9{\cdot}812581} \\ &\quad\; {\cdot}000037. \end{aligned}$$

Assuming that the increase of the logarithm is proportional to the increase in the angle, we have, taking x as the required increase in the logarithm—

$$x : {\cdot}000037 :: 9'' : 15'';$$

Therefore $x = \frac{9}{15} \, {\cdot}000037 = {\cdot}000022.$

Thus L sin. 40° 30′ 24″ $= 9{\cdot}812581 + {\cdot}000022 = 9{\cdot}812603.$

116. *To find the angle which corresponds to a given tabular logarithmic sine.*

Thus let it be required to find the angle which has for the tabular logarithm of its sine 9·688723.

Reference to the tables shows that the given tabular logarithm lies between those of the sines of 29° 13′ 45″ and 29° 14′.

$$\begin{aligned} \text{Thus L sin. } 29^\circ\ 14'\ \ 0'' &= 9{\cdot}688747 \\ \text{L sin. } 29\ \ 13\ \ 45\ &= \underline{9{\cdot}688690} \\ \text{difference } &= \ \ {\cdot}000057 \end{aligned}$$

$$\begin{aligned} \text{L sine of angle required} &= 9{\cdot}688723 \\ \text{L sin. } 29^\circ\ 13'\ 45''\ \ .\ \ . &= \underline{9{\cdot}688690} \\ \text{difference } &= \ \ {\cdot}000033 \end{aligned}$$

Hence if x be the excess of the required angle above 29° 13′ 45″, we have

$$x : 15'' :: {\cdot}000033 : {\cdot}000057.$$

Therefore $x = \frac{33}{57}\, 15'' = 9''$ nearly.

And the angle required = 29° 13′ 45″ + 9″ = 29° 13′ 54″.

117. *To find the tabular logarithmic cosine of a given angle.*

Let it be required to find the tabular logarithmic cosine of the angle 47° 44′ 21″.

We have from the tables

$$\begin{aligned} \text{L cos. } 47^\circ\ 44'\ 15'' &= 9{\cdot}827711 \\ \text{L cos. } 47\ \ 44\ \ 30\ &= \underline{9{\cdot}827676} \\ \text{difference } &= \ \ {\cdot}000035 \end{aligned}$$

Since in the first quadrant the cosine decreases as the angle increases, let x be the required decrease in the tabular logarithm, corresponding to the increase of 6″ in the angle, from 47° 44′ 15″, namely, to 47° 44′ 21″.

Then, making use, as before, of the principle of proportion, we have

$$x : {\cdot}000035 :: 6'' : 15'';$$

therefore $x = \frac{2}{5}\, {\cdot}000035 = {\cdot}000014.$

Therefore the tabular logarithm required is 9·827711 $-$·000014 = 9·827697.

118. *To find the angle corresponding to a given tabular logarithmic cosine.*

Suppose the given tabular logarithm to be 9·864532.

The tables give us

L cos. 42° 56′ 30″	. .	= 9·864539
L cos. 42 56 45	. .	= 9·864510
difference	.	= ·000029
L cos. 42° 56′ 30″	. .	= 9·864539
L cosine of angle required		= 9·864532
difference	.	= ·000007.

Thus we have to find x, the increase in the angle corresponding to ·000007, the decrease in the tabular logarithm.

Therefore $x = \frac{7}{29}\, 15'' = 4''$ nearly.

And the angle sought for is 42° 56′ 30″ + 4″ = 42° 56′ 34″.

119. It is unnecessary to give further examples. For any of the six functions the tabular logarithms of angles intermediate in value between those given in the tables may be found by applying the principle of proportion, and if we have given the logarithm we may determine the value of the angle to which it belongs by the same method. It must, of course, be remembered that in the first quadrant the tangent and secant increase with the increase of the angle, while the cosecant and cotangent decrease within these limits.

120. It should be observed in connection with these tables that the tabular logarithmic sines are given for each second of angle up to 50′. The reason for this is to be found in the fact that the sines of small angles vary with great rapidity, so that the ordinary method of proportioning for seconds would lead to erroneous results.

121. Of a similar character to those lately described is the table containing what are called the logarithmic haversines, which occupies a prominent position in Inman's tables. It is

of considerable importance in connection with what is known as the 'solution of triangles,' as will appear in the next chapter.

The word haversine is a contraction of half-versine. The versine has been already defined to be the defect of the cosine from unity.

Thus $\text{hav. A} = \dfrac{\text{vers. A}}{2} = \dfrac{1-\text{cos. A}}{2} = \dfrac{2\ \text{sin.}^2 \frac{\text{A}}{2}}{2} = \text{sin.}^2 \dfrac{\text{A}}{2}$.

The haversine of an angle is therefore the square of the sine of half the angle, and the logarithmic haversines might be deduced from the table of logarithmic sines. They are, however, of constant use in practical calculations, and it is convenient that they should be given separately.

Since the value of the versine increases continually from 0° to 180°, the tabular logarithmic haversines are given for all angles up to this limit. To 135° the logarithms are set down for each 15″; as the angle approaches 180°, however, the haversine changes very slowly, and it is found sufficient to give the logarithm for a change of each minute of angle.

The tabular logarithmic haversines for values of the angle intermediate in value between those given in the tables may be found by applying the principle of proportion, as in the cases of the sine, cosine, &c.

122. On account of the great simplification which the use of logarithms introduces into the several processes of calculation, it is in general more important to have the tables of the logarithms of the sine, cosine, &c. of an angle than the actual values of the functions themselves. In cases where we have need of the actual value of one of the ratios, as, for instance, of the sine, or, as it is called, the *natural sine*, we may easily deduce its value from the tabular logarithm by making use of the table of logarithms of numbers.

Thus L sin. 30° = 9·698970, or log. sin. 30° = $\bar{1}$·698970.

Referring to the table of logarithms of numbers, we find this to be the value of ·5, or $\frac{1}{2}$.

123. One case, however, occurs in which the value of the function itself is frequently required. It is that of the *versine.*

The values of the *natural versine* are therefore recorded in a table for every minute of angle from 0° to 180°.

By means of a column of 'proportional parts for seconds,' constructed upon the same principle as those given with the logarithms of numbers, the natural versine of an angle given to the nearest second may be readily determined with great exactness.

Since the value of the versine in general takes a fractional form, it is found advantageous for convenience in printing the tables to multiply each versine by 1000000; in other words, to omit the decimal point.

In order, therefore, to obtain the real value of the versine from that given in the tables, which is called the *tabular versine*, we must first divide by that quantity.

Thus the tabular versine of 60° is given as 500000.

$$\text{Therefore vers. } 60° = \frac{500000}{1000000}, \text{ or } \frac{1}{2}.$$

124. The following example will serve to explain the method of using the table of natural versines :—

Let it be required to find the tabular versine of 78° 16′ 27″.

The value of the tabular versine of 78° 16′ is found directly to be 796643. To find the quantity to be added for 27″, we carry the eye along the horizontal line of figures set on the right-hand page against 27″ until the column headed 78° 0′ is reached, whence is taken the number 128, being the part for 27″.

Then we have

$$\begin{aligned} \text{tabular versine of } 78° \, 16' &= 796643 \\ \text{part for } 27'' &= \underline{\quad\;\; 128} \\ \text{tabular versine of } 78° \, 16' \, 27'' &= 796771. \end{aligned}$$

Had the number of minutes in the given angle been 30, or greater than 30, the parts for seconds would have been taken from the vertical column headed 78° 30′.

125. Again, suppose we have given the tabular versine 439672, and that it is required to find the corresponding angle.

We find in the tables that 439602, the tabular versine of 55° 55′, is the next in value below that given. Thus

$$\begin{aligned} \text{tabular versine given} &= 439672 \\ \text{tabular versine of } 55^\circ\, 55' &= \underline{439602} \\ \text{difference} &= \quad\; 70 \end{aligned}$$

This difference must be due to the parts for seconds. Looking out 70 in the column headed 55° 30′, we find that 70 lies midway between the parts for 17 and for 18 seconds. The angle 55° 55′ 18″ may therefore be taken for the value required.

126. A few examples are added here to assist the student in familiarising himself with the arrangement of the several tables.

In Part III., the practical portion of this treatise, ample opportunity will be afforded of acquiring proficiency in the ready and accurate practical use of logarithms and logarithmic tables.

EXAMPLES.—XXI.

The following two logarithms will be required in solving some of the examples :—

Log. 2 = ·301030; log. 3 = ·477121.

1. Deduce from the table of logarithms of numbers the tabular logarithms of sin. 45°, tan. 45°, sin. 60°, cos. 60°, cot. 30°, and tan. 210°.

2. Prove that L tan. A + L cot. A = 20, and that L cos. A + L tan. A + L cosec. A = 30.

3. If L sin. A = 9·700280, and L cos. A = 9·937092, find L tan. A.

4. If L tan. A exceeds L sin. A by ·062762, find the value of L cos. A.

5. Given L sin. $\frac{A}{2}$ = 9·741889, and L cos. $\frac{A}{2}$ = 9·921107, find L cosec. A.

6. If L sec. A + L cosec. A = 20·316086, find L sin. 2A.

7. If L hav. A = 9·301030, find L cos. A.

8. If L cos. A = 9·477121, find the tabular versine of A.

9. If L hav. 2A = 9·847183, find L sin. A.

10. If the tabular versine of A = 1500000, find L hav. A.

11. If L hav. A = 9·000000, find the tabular versine of A.

12. The difference between the tabular versines of two angles is 300000, and the sum of their tabular logarithmic cosines is 19; find the natural cosines of the angles.

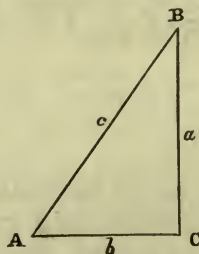
CHAPTER XVI.

ON THE FORMULÆ FOR THE SOLUTION OF TRIANGLES.

127. WE have now reached a point where the actual value of trigonometry, considered as a practical science, becomes manifest. One of its principal objects, as its name implies, was to establish certain relations between the sides and angles of triangles, so that when some of these are known the rest may be determined. Certain relations of this kind we have already from geometry. We shall proceed to determine others.

128. Every triangle has six *parts*, as they are termed, viz. three sides and three angles, some of which being given we are able to determine the remainder.

The simplest cases which occur are those which deal with *right-angled* triangles. These will therefore be first treated.

129. Let ABC be a triangle having the angle at C a right angle. Let the letters a, b, c be the measures of the sides opposite to the angles A, B, C respectively.

130. *In any right-angled triangle each side is equal to the*

product of the hypothenuse into the sine of the opposite angle, or is equal to the product of the hypothenuse into the cosine of the adjacent angle.

By definition, $\frac{BC}{AB} = \frac{a}{c} = \sin. A$;

$\therefore \quad a = c \sin. A.$

Again, by definition, $\frac{AC}{AB} = \frac{b}{c} = \cos. A$;

$\therefore \quad b = c \cos. A.$

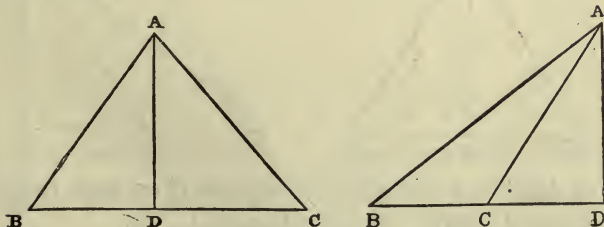
131. In the same way it may be shown that $a = b \tan. A$, or $= b \cot. B$; or, as it may be stated in words: *In any right-angled triangle each side is equal to the product of the tangent of the opposite angle into the other side, or is equal to the product of the cotangent of the adjacent angle into the other side.*

Generally it will be found that one side of a right-angled triangle may always be expressed in the form of a product, in which one of the other sides is multiplied by some function of one of the angles of the triangle.

Collecting these results, we have

$a = c \sin. A$ or $= c \cos. B$ or $= b \tan. A$ or $= b \cot. B$;
$b = c \cos. A$ or $= c \sin. B$ or $= a \tan. B$ or $= a \cot. A$;
$c = a \operatorname{cosec}. A$ or $= a \sec. B$ or $= b \operatorname{cosec}. B$ or $= b \sec. A$.

132. The formulæ now to be established are true for all plane triangles, right-angled or otherwise.

133. *In any triangle the sides are proportional to the sines of the opposite angles.*

Let ABC be any triangle, and from A draw AD perpendicular to the opposite side, meeting that side, or side produced,

in D. If B and C be both *acute* angles, we have from the left-hand figure

$$AD = AB \sin. B, \text{ and } AD = AC \sin. C.$$

Therefore $AB \sin. B = AC \sin. C.$

$$\therefore \quad \frac{c}{b} = \frac{\sin. C}{\sin. B}.$$

If one of the angles, as C, be obtuse, we have from the right-hand figure

$$AD = AB \sin. B., \text{ and } AD = AC \sin. (180° - C) = AC \sin. C.$$

Therefore $AB \sin. B = AC \sin. C.$

$$\therefore \quad \frac{c}{b} = \frac{\sin. C}{\sin. B}.$$

If the angle C be a right angle, we have from the figure of art. 129

$$AC = AB \sin. B;$$

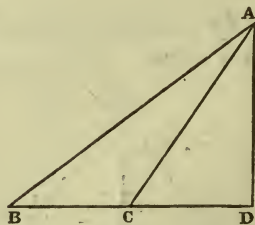
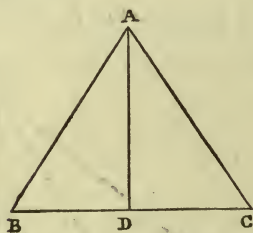
$$\therefore \quad \frac{c}{b} = \frac{1}{\sin. B} = \frac{\sin. C}{\sin. B}$$

Similarly it may be shown that

$$\frac{a}{b} = \frac{\sin. A}{\sin. B}, \text{ and } \frac{a}{c} = \frac{\sin. A}{\sin. C}.$$

These results may be written thus:

$$\frac{\sin. A}{a} = \frac{\sin. B}{b} = \frac{\sin. C}{c}.$$

134. *To express the cosine of an angle of a triangle in terms of the sides.*

Let ABC be a triangle, and suppose C an *acute* angle, as in the left-hand figure. Then by Euclid, II. 13,

$$AB^2 = BC^2 + AC^2 - 2BC \,.\, CD,$$

$$\text{and } CD = AC \cos. C;$$

$$\therefore \quad c^2 = a^2 + b^2 - 2ab \cos. C.$$

Next suppose C an *obtuse* angle, as in the right-hand figure. Then by Euclid, II. 12,

$$AB^2 = BC^2 + AC^2 + 2BC \cdot CD,$$

$$\text{and } CD = AC \cos.(180° - C) = -AC \cos. C;$$

$$\therefore \qquad c^2 = a^2 + b^2 - 2ab \cos. C.$$

Thus in both cases

$$\cos. C = \frac{a^2 + b^2 - c^2}{2ab}.$$

If C be a right angle, cos. C = 0, so that the formula $c^2 = a^2 + b^2 - 2ab$, cos. C becomes $c^2 = a^2 + b^2$, which is true by Euclid, I. 47.

The relation, therefore, holds, whatever may be the value of the angle C.

Similarly it may be shown that

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc};$$

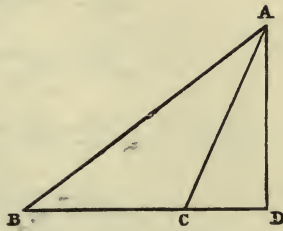
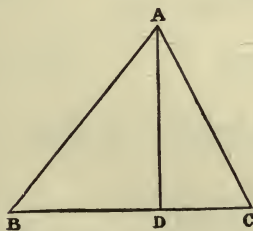
$$\cos. B = \frac{c^2 + a^2 - b^2}{2ca}.$$

135. *In every triangle each side is equal to the sum of the product of each of the other sides into the cosine of the angle which it makes with the first side.*

If the angle C be *acute*, we have from the left-hand figure

$$BC = BD + DC = AB \cos. B + AC \cos. C.$$

Therefore $a = c \cos. B + b \cos. C$.

If the angle C be obtuse, we have from the right-hand figure

$$BC = BD - DC = AB \cos. B - AC \cos.(180° - C)$$
$$= AB \cos. B + AC \cos. C.$$

Therefore $a = c \cos. B + b \cos. C$.

Similarly we shall have

$$b = a \cos. C + c \cos. A, \text{ and } c = b \cos. A + a \cos. B.$$

136. The value of the cosine of an angle in terms of the sides may be made to depend on this property. Thus

$$b = a \cos. C + c \cos. A \quad \therefore \quad b^2 = ab \cos. C + bc \cos. A$$
$$c = a \cos. B + b \cos. A \quad \therefore \quad c^2 = ac \cos. B + bc \cos. A$$
$$a = b \cos. C + c \cos. B \quad \therefore \quad a^2 = ab \cos. C + ac \cos. B.$$

By adding each side of the first two equations, and from the sum subtracting the third equation, we obtain

$$b^2 + c^2 - a^2 = 2bc \cos. A;$$
$$\therefore \cos. A = \frac{b^2 + c^2 - a^2}{2bc};$$

and in the same way the values for cos. B and cos. C may be derived.

137. *To express the sine, the cosine, and the tangent of half of an angle of a triangle in terms of the sides.*

To show that

$$\sin. \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ where } s = \frac{a+b+c}{2}.$$

Since $\cos. A = 1 - 2 \sin.^2 \frac{A}{2}$,

$$\therefore \quad 2 \sin.^2 \frac{A}{2} = 1 - \cos. A$$
$$= 1 - \frac{b^2 + c^2 - a^2}{2bc} \text{ (art. 134)}$$
$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$
$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$
$$= \frac{a^2 - (b-c)^2}{2bc}$$
$$= \frac{(a+b-c)(a-b+c)}{2bc}.$$

Let $a + b + c = 2s$.

Then $a + b - c = 2s - 2c$, and $a - b + c = 2s - 2b$;

therefore $$2 \sin.^2 \frac{A}{2} = \frac{2(s-c)\,2(s-b)}{2bc};$$

and $$\sin. \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

And the sign to be affixed to the root symbol must be always positive, since A, being an angle of a triangle, must be less than 180°, and therefore $\frac{A}{2}$ less than 90°.

Next to show that

$$\cos. \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \text{ where } s = \frac{a+b+c}{2}.$$

Since $$\cos. A = 2 \cos.^2 \frac{A}{2} - 1,$$

therefore $$2 \cos.^2 \frac{A}{2} = \cos. A + 1$$

$$= \frac{b^2 + c^2 - a^2}{2bc} + 1$$

$$= \frac{b^2 + 2bc + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc}.$$

But if $a + b + c = 2s$, $b + c - a = 2s - 2a$;

therefore $$2 \cos.^2 \frac{A}{2} = \frac{2s \,.\, 2(s-a)}{2bc}.$$

Therefore $$\cos. \frac{A}{2} = \sqrt{\frac{s \,.\, (s-a)}{bc}}.$$

From the values of $\sin. \frac{A}{2}$, and $\cos. \frac{A}{2}$, we obtain

$$\tan. \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

And, as in the case of the sine, since $\frac{A}{2}$ is always less than 90°, the values of $\cos. \frac{A}{2}$ and $\tan. \frac{A}{2}$ must always be positive.

138. *To express the haversine of an angle in terms of the sides.*

Since $$\cos. A = \frac{b^2+c^2-a^2}{2bc},$$

$$\therefore \quad 1-\cos. A = 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{a^2-b^2+2bc-c^2}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc}.$$

But $1-\cos. A = \text{vers. } A = 2 \text{ hav. } A$;

$$\therefore \quad 2 \text{ hav. } A = \frac{a^2-(b-c)^2}{2bc};$$

$$\therefore \quad \text{hav. } A = \frac{a^2-(b-c)^2}{4bc} = \frac{2(s-b)\,2(s-c)}{4bc} = \frac{(s-b)(s-c)}{bc}.$$

Since, as pointed out in art. 121, hav. $A = \sin.^2 \frac{A}{2}$, the value here obtained for hav. A might have been deduced from that found for sin. $\frac{A}{2}$ in the preceding article. The expression for the haversine, however, is of very great importance in the solution of triangles, and the proof is therefore exhibited separately.

139. *To express the sine of an angle of a triangle in terms of the sides.*

$$\text{Sin. } A = 2 \sin. \frac{A}{2} \cos. \frac{A}{2}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

140. *To show that*

$$\tan. \frac{A-B}{2} = \frac{a-b}{a+b} \cot. \frac{C}{2}.$$

Since $$\frac{a}{b} = \frac{\sin. A}{\sin. B},$$

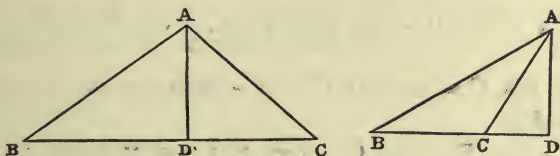
therefore $$\frac{a-b}{a+b} = \frac{\sin. A - \sin. B}{\sin. A + \sin. B}$$

$$= \frac{2 \cos. \frac{A+B}{2} \sin. \frac{A-B}{2}}{2 \sin. \frac{A+B}{2} \cos. \frac{A-B}{2}}$$

$$= \tan. \frac{A-B}{2} \cot. \frac{A+B}{2}$$

$$= \tan. \frac{A-B}{2} \tan. \frac{C}{2}, \text{ since } \frac{A+B}{2} = 90^\circ - \frac{C}{2}.$$

Therefore $$\tan. \frac{A-B}{2} = \frac{a-b}{a+b} \cot. \frac{C}{2}.$$

141. *To show that the area of a triangle is equal to one-half the product of two sides multiplied by the sine of the angle included between those sides.*

Let ABC be any triangle, and AD the perpendicular upon the side BC.

Since a triangle is equal to one-half the rectangle upon the same base, and of the same altitude.

The area of ABC $= \frac{1}{2}$ BC . AD $= \frac{1}{2}\, a.c \sin. B$, or $\frac{1}{2}\, a.b \sin. C$.

In the same way the area may be shown to $= \frac{1}{2}\, b.c \sin. A$.

142. *To express the area of a triangle in terms of the sides.*

Since the area of triangle ABC $= \frac{1}{2}\, bc \sin. A$ (art. 141).

But, by art. 139, $\sin. A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$.

Therefore the area of ABC $= \sqrt{s(s-a)(s-b)(s-c)}$.

This is denoted sometimes by the symbol S, so that

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

143. Since the three angles of a triangle are equal to 180°, the following results will hold:—

(1) $\sin.(A + B) = \sin. C.$

(2) $\cos.(A + B) = -\cos. C.$

(3) $\sin.\frac{A + B}{2} = \cos.\frac{C}{2}.$

(4) $\cos.\frac{A + B}{2} = \sin.\frac{C}{2}.$

144. By the employment of suitable artifices a large number of relations may be established between the angles and sides of a triangle in addition to those already given.

Thus, let it be required to show that, if A, B, C be the angles of a triangle,

$$\cos. A + \cos. B - \cos. C = 4\cos.\frac{A}{2}\cos.\frac{B}{2}\sin.\frac{C}{2} - 1.$$

Since $\cos. C = \cos.(180° - A - B) = -\cos.(A + B)$,
Therefore

$$\cos. A + \cos. B - \cos. C = \cos. A + \cos. B + \cos.(A + B)$$
$$= 2\cos.\frac{A + B}{2}\cos.\frac{A - B}{2} + 2\cos.^2\frac{A + B}{2} - 1$$
$$= 2\cos.\frac{A + B}{2}\left(\cos.\frac{A - B}{2} + \cos.\frac{A + B}{2}\right) - 1$$
$$= 2\cos.\frac{A + B}{2}\left(2\cos.\frac{A}{2}\cos.\frac{B}{2}\right) - 1$$
$$= 4\sin.\frac{C}{2}\cos.\frac{A}{2}\cos.\frac{B}{2} - 1.$$

EXAMPLES.—XXII.

1. In any plane triangle ABC establish the following relations:—

(1) $\sin. A + \sin. B + \sin. C = 4\cos.\frac{A}{2}\cos.\frac{B}{2}\cos.\frac{C}{2}.$

(2) $\sin. A - \sin. B + \sin. C = 4\sin.\frac{A}{2}\cos.\frac{B}{2}\sin.\frac{C}{2}.$

(3) $\sin.^2 A + \sin.^2 B + \sin.^2 C - 2\cos. A\cos. B\cos. C = 2.$

(4) $\text{Cot.}\,\frac{A}{2} + \cot.\,\frac{B}{2} + \cot.\,\frac{C}{2} = \cot.\,\frac{A}{2}\cot.\,\frac{B}{2}\cot.\,\frac{C}{2}.$

(5) $\text{Tan.}\,\frac{A}{2}\tan.\,\frac{B}{2} + \tan.\,\frac{B}{2}\tan.\,\frac{C}{2} + \tan.\,\frac{C}{2}\tan.\,\frac{A}{2} = 1.$

(6) $\dfrac{\sin.\,A + \sin.\,B - \sin.\,C}{\sin.\,A + \sin.\,B + \sin.\,C} = \tan.\,\frac{A}{2}\tan.\,\frac{B}{2}.$

(7) $\text{Cot.}\,A + \cot.\,B + \cot.\,C = \cot.\,A\cot.\,B\cot.\,C + \text{cosec.}\,A\,\text{cosec.}\,B\,\text{cosec.}\,C.$

(8) $\text{Cos.}\frac{A}{2} + \cos.\frac{B}{2} + \cos.\frac{C}{2} = 4\cos.\frac{B+C}{4}\cos.\frac{A+C}{4}\cos.\frac{A+B}{4}.$

(9) $\text{Sin.}\,2A + \sin.\,2B + \sin.\,2C = 4\sin.\,A\sin.\,B\sin.\,C.$

(10) $\text{Cos.}\,2A + \cos.\,2B + \cos.\,2C + 4\cos.\,A\cos.\,B\cos.\,C + 1 = 0.$

2. Show that if in the triangle ABC the angle C is a right angle, the following relations hold :—

(1) $\text{Cot.}\,\frac{A}{2} = \frac{b+c}{a}.$

(2) $\text{Sin.}\,\frac{B}{2} = \sqrt{\frac{c-a}{2c}}.$

(3) $\dfrac{\text{Cos.}\,2B - \cos.\,2A}{\sin.\,2A} = \tan.\,A - \tan.\,B.$

(4) $\text{Sec.}\,2A = \dfrac{c^2}{b^2 - a^2}.$

3. In any plane triangle ABC prove the following relations :—

(1) $a\sin.\,\frac{1}{2}(B - C) = (b - c)\cos.\,\frac{A}{2}.$

(2) $a\cos.^2\frac{B}{2} + b\cos.^2\frac{A}{2} = b\cos.^2\frac{C}{2} + c\cos.^2\frac{B}{2}$

$= c\cos.^2\frac{A}{2} + a\cos.^2\frac{C}{2}.$

(3) $a\sin.\,(B - C) + b\sin.\,(C - A) + c\sin.\,(A - B) = 0.$

(4) $\dfrac{\sin.\,A}{\sin.\,B} = \dfrac{\cos.\,A\cos.\,C + \cos.\,B}{\cos.\,B\cos.\,C + \cos.\,A}.$

(5) $\left(\frac{b}{c} + \frac{c}{b}\right)\cos.\,A + \left(\frac{c}{a} + \frac{a}{c}\right)\cos.\,B + \left(\frac{a}{b} + \frac{b}{a}\right)\cos.\,C = 3.$

(6) $\text{Sin. } 2A + \sin. 2B + \sin. 2C$
$= 4 \cos. A \cos. B \cos. C (\tan. A + \tan. B + \tan. C)$.

(7) $\text{Cot. } \frac{A}{2} = \frac{\cos. C - \cos. B}{\sin. B - \sin. C}$.

(8) $(s-a) \tan. \frac{A}{2} = (s-b) \tan. \frac{B}{2} = (s-c) \tan. \frac{C}{2}$.

(9) If p, q, r be the perpendiculars from A, B, C upon the opposite sides, then

$$p \sin. A = q \sin. B = r \sin. C.$$

(10) $(a^2 - b^2) \sin. A = ac \sin. (A-B)$.

(11) $\frac{1 + \cos. (A-B) \cos. C}{1 + \cos. (A-C) \cos. B} = \frac{a^2 + b^2}{a^2 + c^2}$.

(12) If A : B : C :: 2 : 3 : 4, then

$$2 \cos. \frac{A}{2} = \frac{a+c}{b}$$

(13) $4 \left(bc \sin.^2 \frac{A}{2} + ca \sin.^2 \frac{B}{2} + ab \sin.^2 \frac{C}{2}\right) + 2 (a^2 + b^2 + c^2)$
$= (a+b+c)^2$.

(14) If $2a = b + c$, $\tan. \frac{B}{2} \tan. \frac{C}{2} = \frac{1}{3}$.

4. Show that the area of a triangle is given by each of the following expressions :—

(1) $bc \sqrt{\text{hav. } A \text{ hav. } (B+C)}$.

(2) $\frac{1}{4} (a+b+c)^2 \tan. \frac{A}{2} \tan. \frac{B}{2} \tan. \frac{C}{2}$.

(3) $\frac{a^2}{2 \sin. A} (\cos. B \cos. C + \cos. A)$.

(4) $\frac{\text{Sin. } A}{2 \cos.^2 A} (c - a \cos. B) (b - a \cos. C)$.

5. If two triangles have angles A, B, C, and $90° - \frac{A}{2}$, $90° - \frac{B}{2}$, $90° - \frac{C}{2}$ respectively, and a common side a opposite to the first-

named angle in each, show that their areas will be in the proportion of 2 sin. $\frac{B}{2}$ sin. $\frac{C}{2}$ to sin. $\frac{A}{2}$.

6. Show that the length of the straight line joining the point A with the middle point of the side BC of the triangle ABC is equal to $\frac{1}{2}\sqrt{b^2+c^2+2bc \cos. A}$.

7. AD is the perpendicular from the vertex A on the base BC of a plane triangle ABC, and E, F are the middle points of AD, BC respectively; prove that

(1) Cot. EFC = cot. B − cot. C.

(2) If $EF = \frac{c}{2}$, then either $a^2=3(b^2-c^2)$, or the triangle is right-angled.

8. Two plane triangles with a common angle A, and the sides containing it b, c, and b', c' respectively, have equal areas; show that two other triangles, which have each an angle A, and have sides containing it b, b' and c, c' respectively, are equiangular.

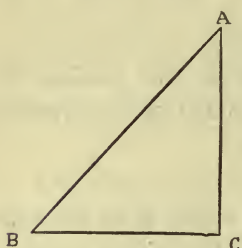
9. If the sum of the squares of the cosines of the angles be equal to unity, show that the triangle is right-angled.

CHAPTER XVII.

ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

145. The solution of triangles is the process by which when a sufficient number of the six parts of any triangle are given the rest may be determined.

It will be seen that of these six parts three must always be given, and of these at least one must be a side. This is obvious from the consideration that if two sides of a triangle be produced, and lines be drawn, parallel to the base, to meet the sides so produced, an indefinite number of triangles will be obtained having for their angles the same values as in the original triangle. In this case therefore we cannot determine the lengths of the sides, but only their ratios to one another.

146. Since the three angles of a triangle are together equal to two right angles, if two angles A, B of the triangle ABC be given, the value of the third angle may be obtained by subtracting the sum of A and B from 180°.

147. Let ABC be a right-angled triangle, having the angle C a right angle.

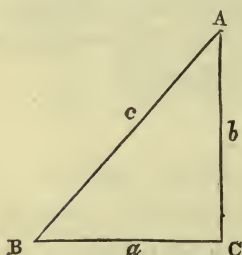
Since in the right angle one part is always given, it will be necessary that only two other parts should be given. The several cases that may occur divide themselves into two heads, according as we have given—

i. Two sides as a, b.

ii. One side and one angle as a, A.

148. *To solve a right-angled triangle, having given two sides.*

Let us suppose that the two sides a, b are given.

The third side might at once be found from the well-known relation between the sides of a right-angled triangle $c^2 = a^2 + b^2$.

The employment of this formula, giving, as it does, the value of c^2 in the form of the *sum* of the squares of the two known quantities, would deprive us of the advantage to be derived from the use of logarithms: such a form should therefore in general be avoided.

It will commonly be more advantageous first to determine one of the unknown angles. Thus $\tan. A = \frac{a}{b}$;

$$\text{therefore } \log. \tan. A = \log. a - \log. b,$$

$$\text{and } L \tan. A = 10 + \log. a - \log. b.$$

Having found the value of A from the tables, since A + B $= 90°$, to obtain B we have only to subtract A from 90°.

To find c we have, by art. 131,

$$c = a \text{ cosec. } A.$$

$$\log. c = \log. a + \log. \text{cosec. } A = \log. a + L \text{ cosec. } A - 10.$$

149. If the parts given be one side and one angle, as a, A, we may at once find the value of the third angle B by subtracting the sum of A, C from 180°. To find the sides b, c we have, by art. 131,

$$b = a \text{ cot. A};$$
$$c = a \text{ cosec. A};$$

whence the values of the two sides may be determined as already explained.

CHAPTER XVIII.

ON THE SOLUTION OF TRIANGLES OTHER THAN RIGHT-ANGLED.

150. In the solution of triangles other than right-angled, four cases present themselves, according as we have given—

(I.) Three sides a, b, c.

(II.) Two angles and one side, as A, B, a or A, B, c.

(III.) Two sides, and the angle opposite to one of those sides, as a, b, A.

(IV.) Two sides and the included angle, as a, b, C.

The several cases will be considered in order.

Case I.

151. *Given the three sides a, b, c to solve the triangle.*

The parts required are the three angles of the triangle.

In art. 134 we have established the formula

$$\text{cos. A} = \frac{b^2 + c^2 - a^2}{2bc}.$$

This formula may sometimes be adopted in simple cases.

Thus let $a = 5$, $b = 4$, $c = 7$.

Then we have $\text{cos. A} = \frac{16 + 49 - 25}{56} = \frac{40}{56} = \frac{5}{7}$.

Therefore L cos. A $= 10 + \text{log. } 5 - \text{log. } 7$;

$$\begin{aligned} 10 + \text{log. } 5 &= 10{\cdot}698970 \\ \text{log. } 7 &= {\cdot}845098 \\ \hline \text{L cos. A} &= 9{\cdot}853872 \end{aligned}$$

whence A $= 44° \; 24' \; 55''$.

152. In practice, however, the custom of resorting at once to logarithms is almost universal; we must therefore have recourse to one of the logarithmic expressions, deduced from the fundamental formula, viz.—

$$\text{Sin. } \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}};$$

$$\text{cos. } \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}};$$

$$\text{tan. } \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

$$\text{hav. } A = \frac{(s-b)(s-c)}{bc}.$$

153. If the student is obliged to confine himself to the use of the ordinary tables of logarithmic sines, cosines, tangents, &c. he may make use of the expression for sin. $\frac{A}{2}$ or cos. $\frac{A}{2}$. It is of little importance whether one or other of these is employed, but when the value of the angle is required to the nearest second, it may be observed that the logarithm of sin. $\frac{A}{2}$ ($\frac{A}{2}$ being always less than 90°) will increase as the angle increases, whereas the logarithm of the cosine decreases as the angle increases. The process of obtaining by proportion the exact number of seconds is therefore slightly less troublesome for sin. $\frac{A}{2}$. In the practical solution of triangles it is in general sufficient to take out angles only to the nearest 15″, so that it is immaterial which formula is employed.

154. If, however, as in the case of Inman's collection, a table of logarithmic haversines is at hand, the solution of the triangle is somewhat simplified by the adoption of the haversine formula, since in the first place we avoid the root symbol on the right-hand side of the equation, and in the second place we obtain at once the value of the angle A, instead of that of half the angle, as in the other cases.

Reduced to logarithms the formula will appear—

$$\text{L hav. } A = 10 + \log. (s-b) + \log. (s-c) - \log. b - \log. c.$$

Thus we obtain the angle A. To obtain B we may, if we please, employ the formula

$$\text{Sin. B} = \frac{b}{a} \text{ sin. A.}$$

In that case, however, any error which may have arisen in the calculation of the angle A will vitiate our results for B also. It is better, therefore, to calculate B independently by the same formula already employed for A. Thus—

$$\text{L hav. B} = 10 + \text{log. } (s-a) + \text{log. } (s-c) - \text{log. } a - \text{log. } c.$$

Having now the values of A and B, by subtracting the sum from 180° we obtain C. When, however, accuracy is of great importance, it may be advisable, by a third application of the haversine formula, to determine C also independently.

We may then test the accuracy of our results by adding the three angles together, when the sum should, of course, be equal to 180°.

Case II.

155. *Given two angles, and one side, to solve the triangle.*

Let the angles given be A, B. Then since $A + B + C = 180°$; C also is known.

First let the side a be given.

Then, since $\frac{b}{a} = \frac{\text{sin. B}}{\text{sin. A}}, \frac{c}{a} = \frac{\text{sin. C}}{\text{sin. A}}$, we have

$$\text{log. } b = \text{log. } a + \text{L sin. B} - \text{L sin. A.}$$
$$\text{log. } c = \text{log. } a + \text{L sin. C} - \text{L sin. A.}$$

If the parts given be A, B, c, the process will be similar to that given above. In each case we have all the angles and one side given, and to find the remaining sides we have only to apply the sine formula as before.

156. It may be observed that the equation $b = \frac{a \text{ sin. B}}{\text{sin. A}}$ may be written $b = a \text{ sin. B cosec. A}$, so that

$$\text{log. } b = \text{log. } a + \text{L sin. B} + \text{L cosec. A} - 20;$$

a formula which is perhaps slightly less troublesome than the other, the process of subtraction being changed into addition.

Case III.

157. *Given two sides and the angle opposite to one of them.*

Let the parts given be the sides a, b, and the angle A.

The consideration of this case divides itself into two parts according as the side a, which is opposite to the angle A, is *greater* or *less* than b.

First let the side a be greater than b. For instance, let us suppose the parts given to be $a = 72$, $b = 64$, $A = 61°$.

We have, from the sine formula,

$$
\begin{aligned}
\text{L sin. B} &= \text{L sin. A} + \log. b - \log. a \\
\text{L sin. A} &= 9{\cdot}941819 \\
\log. b &= \underline{1{\cdot}806180} \\
\text{sum.} &= 11{\cdot}747999 \\
\log. a &= \underline{1{\cdot}857333} \\
\text{difference} = \text{L sin. B} &= 9{\cdot}890666
\end{aligned}
$$

Now, when all that we know of an angle is its sine, since sin. A = sin. (180° − A), there will be two angles less than 180° possessing this sine, and we may be in doubt which of these to select.

Thus, in the present instance, 9·890666 is the tabular logarithmic sine corresponding to each of the angles 51° 1′ 36″, and 128° 58′ 24″.

Since, however, the greater angle of every triangle must be opposite to the greater side, the value 128° 58′ 24″ is inadmissible, being greater than 61°, which is the value of the angle given as subtending the greater of the two sides.

There is, therefore, no ambiguity, and the value 51° 1′ 36″ is clearly that which we seek.

158. Secondly, let the same parts be given, viz. a, b, and A, but let a be less than b.

As in the previous case, we have

$$\text{L sin. B} = \text{L sin. A} + \log. b - \log. a.$$

From this formula, as before, we may obtain two values of B, each less than 180°, and each greater than A.

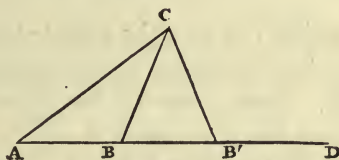
Each of these values may therefore be retained consistently

with the necessary condition that the greater angle shall be opposite to the greater side.

This is an instance of what is known as 'the ambiguous case' in the solution of triangles, and, as will be seen, two triangles may sometimes be found which satisfy the given conditions.

As a simple case, let $a = 1$, $b = \sqrt{3}$, $A = 30°$.

Take any straight line $AC = \sqrt{3}$ units of length.

At A make the angle $CAD = 30°$.

With centre C, and distance equal to the unit of length, describe a circle.

This will cut AD in B, B′.

We have now two triangles, ABC, AB′C, which satisfy the given conditions, viz. that $a = 1$, $b = \sqrt{3}$, $A = 30°$.

To obtain the angles B, B′ we have

$$\sin. B = \frac{b}{a} \sin. 30° = \frac{\sqrt{3}}{2}.$$

Thus B may be either 120°, as in the triangle ABC, or 60°, as in AB′C.

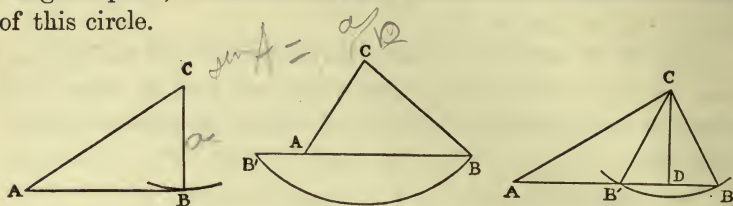
159. The geometrical construction given in the preceding article may be extended to illustrate generally the several cases involved in the solution of a triangle, in which the parts given are the two sides, and the angle opposite to one of these sides.

Thus, having given A, a, b, to construct the triangle ABC.

As before, draw the straight line AC equal to b, and at the point A in CA make the angle CAB equal to A.

With centre C and radius CB, equal to a, describe an arc of a circle.

In order that a triangle may be constructed having the

assigned parts, AB must either touch or cut the circumference of this circle.

First let AB touch the circumference of the circle.

Then ABC is the triangle required, and it has one angle B a right angle.

Next let the radius CB, or a, be greater than b. Then CB will cut AB twice, but one point of intersection will lie upon the left of A, as B′. Thus there will be one triangle only, viz. ABC, which possesses the given parts A, a, b.

Thirdly, let CB be less than CA or b. Then, as shown in the previous article, the circumference of the circle will intersect AB in two points B, B′, upon the same side of A, and two triangles will be formed, ABC, AB′C, each of which has the given values for the angle A and the sides a, b.

160. It should be observed that a must not be less than b sin. A, for in that case CB would be less than CD, so that the construction would fail, and no triangle could be formed having the values given for its several parts.

Case IV.

161. *Given two sides, and the included angle, to solve the triangle.*

Let the parts given be the two sides a, b and the angle C.

We have to determine the two remaining angles A, B, and the third side c.

We may sometimes in simple cases make use of the formula

$$c^2 = a^2 + b^2 - 2ab \cos. C.$$

Then, having the three sides, we may proceed to find the angles as already explained.

In practice, however, it rarely happens that the use of this formula is advantageous, and though by suitable artifices the

expression may be transformed into a logarithmic shape, the process of determining the third side independently by this method is a cumbrous one, and has no advantage to compensate for the amount of trouble involved.

It is better, therefore, to first determine A and B; having then a, A and C, we may easily find c.

It should be observed that A is supposed to be the greater of the two angles.

To obtain A, B.

The sum of these angles is already known, for it is the supplement of the angle C. If, therefore, we can find the difference, we have two simultaneous equations, whence the values of the two quantities A, B may be obtained.

Now by art. 140

$$\tan. \frac{A-B}{2} = \frac{a-b}{a+b} \cot. \frac{C}{2}.$$

Therefore $L \tan. \frac{A-B}{2} = \log. (a-b) + L \cot. \frac{C}{2} - \log. (a+b)$.

This equation gives the value of $\frac{A-B}{2}$, and $\frac{A+B}{2}$ is already known.

The sum of the two values is therefore equal to A, the difference to B.

To find c we have, by art. 133, $\frac{c}{a} = \frac{\sin. C}{\sin. A}$, from which we obtain

$$\log. c = L \sin. C + L \operatorname{cosec}. A + \log. a - 20.$$

162. *To find the area of a triangle, having given two sides and the included angle.*

Let the parts given be a, b, C. It is required to find the area of the triangle.

By art. 141 area $= \frac{1}{2} ab \sin. C$.

Therefore $\log. \text{area} = \log. a + \log. b + L \sin. C - \log. 2 - 10$.

163. *To find the area of a triangle, having given the three sides.*

By art. 142 area $= \sqrt{s(s-a)(s-b)(s-c)}$.

Therefore

$$\log.\ \text{area} = \frac{1}{2}\{\log.\ s + \log.\ (s-a) + \log.\ (s-b) + \log.\ (s-c)\}.$$

Thus in every case where we have sufficient data to solve the triangle we are in a position to find its area.

CHAPTER XIX.

PROBLEMS ON THE SOLUTION OF TRIANGLES.

164. WE shall show in the present chapter the practical application of the processes explained in previous chapters in determining the heights and distances of visible but inaccessible objects.

The following definitions will be required :—

The angle which the line joining the eye of the observer with a distant object makes with the horizontal plane is called the angle of elevation when the object is above the observer.

The same angle is called the angle of depression when the object is below the observer.

By means of a theodolite, angles between objects situated in the same horizontal or vertical plane may be observed; by means of a sextant the angle between objects situated in any plane may be determined.

165. *To find the height of a column standing on a horizontal plane, the base of the column being attainable.*

Let AB be the vertical column.

From the base B measure the horizontal line BC in any direction.

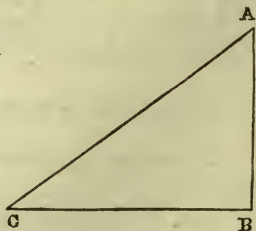
At C observe the angle of elevation ACB.

Then AB = BC tan. ACB.

The line BC is called a *base line.*

The employment of such a base line is very general in problems of this nature, and its careful measurement is essential to accuracy in the final result.

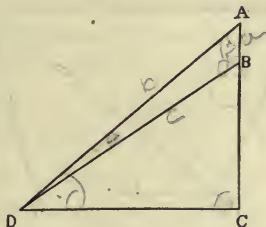
A

C B

166. *To find the height of a flag-staff on the top of a tower.*

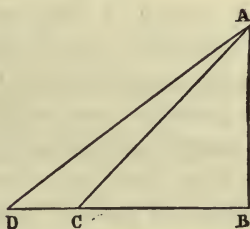
Let AB be the flag-staff standing on the tower BC.

From C measure the base line CD.

At D observe the angles of elevation ADC, BDC.

Then AB = AC − BC = CD tan. ADC − CD tan. BDC.

167. *To find the height of a column standing on a horizontal plane, when the base of the column is inaccessible.*

Let AB be the column.

Measure a distance CD in the same horizontal line with B.

At C, D observe the angles of elevation ACB, ADB.

Then in the triangle ACD we know the angle ACD, since it is the supplement of the angle ACB. Moreover the angle ADC is known, and also the side CD.

Therefore AC may be determined.

Then in the triangle ACB

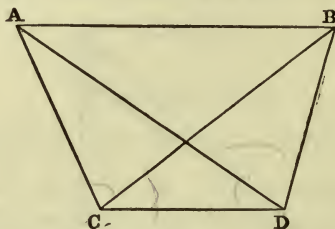
AB = AC sin. ACB.

168. *To find the distance between two inaccessible objects in the same horizontal plane.*

Let A, B be the objects.

Measure a base line CD in the same horizontal plane as **A**, **B**. At C observe the angles ACB, BCD.

At D observe the angles ADC and BDA.

Then in the triangle ADC we know the angles ACD and ADC, and the side CD.

Hence AC may be determined.

In the triangle BDC we have the angles BDC, BCD and the side CD.

Hence BC may be determined.

Then in the triangle ABC, having the two sides AC, BC and the included angle ACB, AB may be obtained.

169.—The practical portion of this work will furnish the student with abundant exercise in the solution of practical problems, involving heights and distances, by the aid of logarithmic tables. In the collection of examples which follows the results required may be established without the aid of logarithms. Many of the examples here given have been proposed in the periodical examinations for the rank of lieutenant held at the Royal Naval College during the past five years.

Examples.—XXIII.

1. From the top of a perpendicular cliff two rocks are observed in a straight line with the base, which are known to be a yards apart; their respective depressions are α and 3α: show that the height of the cliff is $\dfrac{a \sin. 3\alpha}{2 \cos. \alpha}$.

2. A man walking due east along a straight road observes that a certain church tower bears E.N.E.; a mile farther on the bearing of the tower is N.N.E.: show that the shortest distance of the tower from the road is half a mile.

3. The altitudes of the top of a flag-staff, at points due south and due west of it, are observed to be 45° and 30° respectively; show that the height of the flag-staff is half of the distance between the points of observation.

4. The elevation of two clouds to a person in the same line with them is a ; when standing on the shadow of one of them its elevation is $2a$, and the other is vertically over him : show that the heights of the clouds are as $2\cos.^2 a$ to 1.

5. A person wishes to find the distance between two places A and B on opposite sides of a brook. He walks from B to a bridge two miles off. Crossing this he continues his walk 6 miles in the same direction to C, which he knows to be 3 miles from A. If A be 4 miles from the bridge, show that AB = 5·86 miles nearly.

6. The elevation of a tower standing on a horizontal plane is observed; a feet nearer it is found to be 45°; b feet nearer still is the complement of the first angle observed : show that the height of the tower is $\frac{ab}{a-b}$ feet.

7. A column stands in a field which has the form of an equilateral triangle, and subtends angles $\tan.^{-1}\frac{4}{\sqrt{3}}$, $\tan.^{-1}\frac{4}{\sqrt{7}}$, $\tan.^{-1}\frac{4}{7}$ at the three corners; show that the height of the column is equal to the length of a side of the field.

8. A person at the top of a mountain observes the angle of depression of an object in the horizontal plane beneath to be 45°; turning through an angle of 30° he finds the depression of another object in the plane to be 30°: show that the distance between the objects is equal to the height of the mountain.

9. A flag-staff a feet high, on the top of a tower, is seen from a certain point in the horizontal plane on which the tower stands to subtend at the eye of the observer an equal angle θ with that subtended by the tower itself; show that the height of the tower $= a \cos. 2\theta$.

10. A statue which stands upon level ground, on a pedestal 27 feet high, subtends an angle $\tan.^{-1}\frac{7}{24}$ at a point 36 feet

from the foot of the pedestal; show that the height of the statue is 21 feet.

11. A fort is seen from a ship bearing E.N.E.; when the ship has sailed due east 4 miles it bears N.N.E.: show that the distance of the fort is now $\sqrt{16-8\sqrt{2}}$ miles.

12. The elevation of a tower standing on a horizontal plane is observed to be θ; at a station m feet nearer it is $90°-\theta$, and at a point n feet nearer still it is 2θ: show that the height of the tower is $\sqrt{(m+n)^2 - \frac{m^2}{4}}$.

13. On the bank of a river a column 186 feet high supports a statue 31 feet high; the statue, to an observer on the opposite bank, subtends the same angle as a man 6 feet high standing at the foot of the column: show that the breadth of the river is approximately 98·5 feet.

14. An observer at the sea-level notices that the elevation of a certain mountain is a; after walking directly towards it for a distance d along a road inclined at an angle γ to the horizontal, he finds the elevation of the mountain to be β: show that the height of the mountain is $d \sin. a \dfrac{\sin. (\gamma-\beta)}{\sin. (a-\beta)}$.

15. From the top of a tower, which stands near a river, it is observed that the distance of the foot of the tower from the river subtends the same angle as the breadth of the river; the height of the tower being h, and the distance of the tower from the river d: show that the breadth of the river is $\dfrac{h^2+d^2}{h^2-d^2} d$.

16. A tower standing on a horizontal plane is surrounded by a moat, which is equal in width to the height of the tower; a person at the top of another tower, whose height is h, and distance from the moat c, observes that the first tower subtends an angle of 45°: show that the height of the first tower is $\dfrac{h^2+c^2}{h-c}$.

17. A person wishing to determine the length of an inaccessible wall places himself due south of one end, and due west of the other, at such distances that the angle subtended by

the wall is in each case equal to a. If l be the distance between the two stations, show that the length of the wall is $l \tan. a$.

18. A and B are two points in a level field, which is adjacent to a lake, in which are moored two buoys, C and D; A is distant d yards from B and both at A and B the distance CD subtends an angle a: show that if the straight lines AD and BC are inclined at an angle θ, the distance between C and D is either

$$\frac{d \sin. a}{\sin. (\theta + a)} \text{ or } \frac{d \sin. a}{\sin. (\theta - a)} \text{ yards.}$$

19. A flag-staff a feet long stands on the top of a tower b feet high; an observer, whose eye is c feet above the level of the foot of the tower, finds that the staff and the tower subtend equal angles at his eye: prove that the length of the straight line joining his eye to the foot of the tower is

$$b \sqrt{\frac{a+b-2c}{a-b}} \text{ feet.}$$

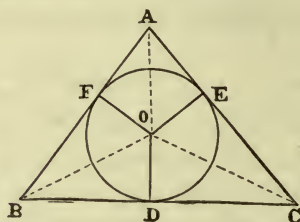
20. The top of a ladder placed against a vertical wall, the foot of which makes an angle of a with the horizon, rests on a window-sill; when its foot is moved m feet farther away from the wall, it makes an angle β with the horizon, and rests on a second sill: show that the distance between the two sills is

$$m \cot. \frac{a+\beta}{2}.$$

21. A column on a pedestal 20 feet high subtends an angle of 45° to a person standing on the level ground upon which the column stands; on approaching 20 feet the observer finds that the angle subtended is again 45°: show that the height of the column is 100 feet.

CHAPTER XX.

OF TRIANGLES AND POLYGONS INSCRIBED IN CIRCLES, AND CIRCUMSCRIBED ABOUT THEM.

170. *To find the radius of the circle inscribed in a triangle.*

Let ABC be a triangle, O the centre of the inscribed circle, which touches the sides BC, CA, AB in D, E, F respectively.

Let r denote the radius of the circle.

Then

$$\text{area of triangle BOC} = \frac{1}{2}\,\text{BC}\,.\,\text{OD} = \frac{ar}{2};$$

$$\text{area of triangle AOC} = \frac{1}{2}\,\text{AC}\,.\,\text{OE} = \frac{br}{2};$$

$$\text{area of triangle AOB} = \frac{1}{2}\,\text{AB}\,.\,\text{OF} = \frac{cr}{2}.$$

Therefore, by addition,

$$\frac{1}{2}(a+b+c)\,r = \text{area of triangle ABC} = \text{S (art. 142)};$$

$$\text{therefore } r = \frac{\text{S}}{s}.$$

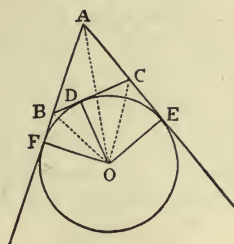
Thus the radius of the inscribed circle is equal to the area of the triangle divided by one-half the sum of the sides.

171. A circle which touches one side of a triangle, and the other two sides produced, is called an *escribed circle.*

172. *To find the radius of an escribed circle.*

Let ABC be a triangle, and let O be the centre of the circle which touches the side BC in D, and AC, AB produced in E, F respectively.

Let r_1 be the radius of this circle.

It will be seen from the figure that the quadrilateral figure

OBAC may be regarded as the sum of the two triangles OAB, OAC.

Therefore the quadrilateral figure OBAC $= \frac{cr_1}{2} + \frac{br_1}{2}$.

Again, the same figure may be divided into the two triangles OBC and ABC.

Therefore the figure OBAC $= \frac{ar_1}{2} + S$.

Hence $$\frac{cr_1}{2} + \frac{br_1}{2} = \frac{ar_1}{2} + S;$$

therefore $$\frac{1}{2}\left(c + b - a\right) r_1 = S;$$

therefore $$r_1 = \frac{S}{s - a}.$$

Similarly if r_2 be the radius of the circle which touches AC and BA, BC produced, and r_3 the radius of the circle which touches AB, and CA, CB produced, it may be shown that

$$r_2 = \frac{S}{s - b}, \ r_3 = \frac{S}{s - c}.$$

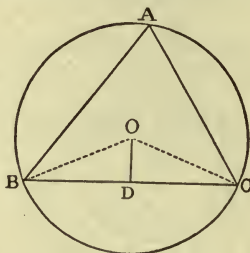
173. *To find the radius of the circle described about a triangle.*

Let ABC be a triangle, and O the centre of the circle described about it.

Draw OD perpendicular to BC. Then BC is bisected at D.

Let R denote the radius of the circle.

Since the angle at the centre of a circle is double of the

angle at the circumference upon the same base, the angle BOC is double of the angle BAC. (Euc. III. 20.)

But the angle BOC is double of the angle BOD.

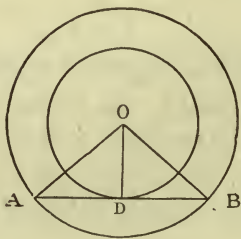
Therefore the angle BOD = the angle BAC.

In the triangle BOD, BD = BO sin. A.

Therefore $\frac{a}{2} = \text{R sin. A}$, and $\text{R} = \frac{a}{2 \text{ sin. A}}$.

Similarly it may be shown that $\text{R} = \frac{b}{2 \text{ sin. B}}$ or $\frac{c}{2 \text{ sin C}}$.

174. *To find the radii of the inscribed and circumscribed circles of a regular polygon.*

Let AB be the side of a regular polygon, that is, of a polygon which has all its sides and angles equal, and let n be the number of sides.

Let O be the centre of the circles, OD the radius of the inscribed, and OA that of the circumscribed circle.

Let AB = a, OA = R, OD = r.

The angle AOB is the nth part of four right angles; therefore AOB = $\frac{2\pi}{n}$, and AOD = $\frac{\pi}{n}$,

and $AD = \frac{a}{2} = R \sin. \frac{\pi}{n} = r \tan. \frac{\pi}{n}$.

Therefore $R = \dfrac{a}{2 \sin. \frac{\pi}{n}}$, $r = \dfrac{a}{2 \tan. \frac{\pi}{n}}$.

The area of the polygon may be expressed by means of the radius of the inscribed circle, or by that of the circumscribed circle.

Thus, the area of the polygon $= n$ times the area of the triangle AOB

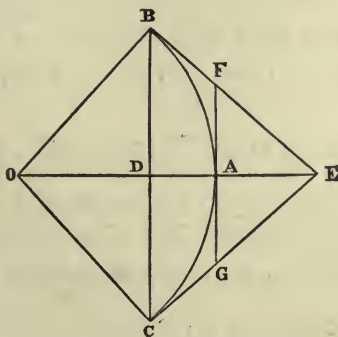
$$= n \,.\, AD \,.\, OD = n \frac{a}{2} r = nr^2 \tan. \frac{\pi}{n}.$$

Again, since $AD = R \sin. \frac{\pi}{n}$, and $OD = R \cos. \frac{\pi}{n}$;

therefore the area of the polygon $= n\, R^2 \sin. \frac{\pi}{n} \cos. \frac{\pi}{n}$

$$= \frac{1}{2} n\, R^2 \sin. \frac{2\pi}{n}.$$

175. *To prove that if θ be the circular measure of a positive angle less than a right angle, θ is greater than sin. θ, but less than tan. θ.*

Let AOB be any angle less than a right angle, and let OB = OA.

From B draw BD perpendicular to OA, and produce it to C, so that DC = DB, and join OC.

Draw BE at right angles to OB, meeting OA produced in E, and join EC.

Then the triangles BOD, COD are equal in all respects, so that the angle BOD is equal to the angle COD.

Therefore the triangles BOE, COE are equal in all respects, the angle ECO is a right angle, and EC = EB.

With centre O, and radius OB, describe an arc of a circle BAC.

This will touch EB at B, and EC at C.

Then if we may assume provisionally that the straight line BC is less than the arc BAC, we have BD, the half of the straight line, less than BA, the half of the arc.

Therefore $\frac{BD}{OB}$ is less than $\frac{BA}{OB}$; that is, the sine of the angle AOB is less than its circular measure.

Again, if we assume that the arc BAC is less than the sum of the two straight lines EB and EC, we have AB less than EB.

Hence $\frac{AB}{OB}$ is less than $\frac{EB}{OB}$; that is, the circular measure of the angle AOB is less than its tangent.

176. With regard to the two assumptions that have been made, the first is sufficiently obvious to be readily admitted.

The second one, namely, that the two straight lines EB, EC are together greater than the arc BAC, requires more consideration.

At the point A in the figure of the preceding article, draw the straight line FG, touching the arc, and meeting EB, EC in F, G.

Then because two sides of a triangle are together greater than the third, it follows that BE, EC are together greater than BF, FG, GC.

Now if the angle BOC $= \frac{2\pi}{n}$, where n is a positive integer, BE, EC will be together equal to the side of a regular polygon of n sides described about the circle of which BAC is an arc.

Again, since FB, FA are tangents drawn to the circle from the same point F.

Therefore FB is equal to FA.

For a similar reason GC is equal to GA.

Therefore BF, FG, GC are together equal to twice FG, that is, they are together equal to *two* sides of a regular polygon of $2n$ sides described about the circle.

Hence the perimeter of the polygon of n sides described about a circle is greater than the perimeter of a polygon of $2n$ sides described about the same circle.

Similarly it may be shown that the perimeter of a polygon of $2n$ sides is greater than the perimeter of a polygon of $4n$ sides, and that generally the perimeter decreases as the number of sides increases.

But since when the number of sides is indefinitely increased the circumscribed polygon approaches the form of a circle, it is clear that the circumference of a circle must be less than the perimeter of any polygon that may be described about it.

And therefore any arc BAC, which is $\frac{1}{n}$th part of the circumference, is less than the sum of EB, EC, which are together $\frac{1}{n}$th part of the perimeter of the circumscribed polygon.

177. *To show that the limit of* $\frac{\sin. \theta}{\theta}$, *when* θ *is indefinitely diminished, is unity.*

Since sin. θ, θ, and tan. θ are in ascending order of magnitude, dividing by sin. θ, we have 1, $\frac{\theta}{\sin. \theta}$, $\frac{1}{\cos. \theta}$ in ascending order of magnitude.

As θ is diminished, cos. θ approaches unity, and the smaller cos. θ becomes, the more nearly does $\frac{1}{\cos. \theta}$ approach unity.

Therefore by diminishing θ sufficiently we can make $\frac{\theta}{\sin. \theta}$ differ from unity by less than any assignable quantity, however small, that is to say, $\frac{\theta}{\sin. \theta}$ will approach the limit unity.

Therefore also $\frac{\sin. \theta}{\theta}$ approaches the limit unity.

Moreover, since $\frac{\tan. \theta}{\theta} = \frac{\sin. \theta}{\theta} \times \frac{1}{\cos. \theta}$, the limit of $\frac{\tan. \theta}{\theta}$, when θ is indefinitely diminished, is also unity.

178. It follows from the preceding article that the limit of $m \sin. \frac{a}{m}$, when m increases indefinitely, is a.

For $m \sin. \frac{a}{m} = \dfrac{a \sin. \frac{a}{m}}{\frac{a}{m}}$, and when m becomes indefinitely great $\dfrac{\sin. \frac{a}{m}}{\frac{a}{m}}$ is unity.

Similarly the limit of $m \tan. \frac{a}{m}$, when m increases indefinitely, is a.

179. *To find the area of a circle.*

By art. 174 the area of a regular polygon of n sides described about a given circle of radius r

$$= nr^2 \tan. \frac{\pi}{n}.$$

Now when n is increased without limit, the figure of the polygon approaches the form of the circle, and therefore the area of the circle will be equal to the limit of the above expression.

But when n is indefinitely great $n \tan. \frac{\pi}{n} = \pi$, by art. 178.

Therefore the area of the circle $= \pi r^2$.

Examples.—XXIV.

1. A square and an equilateral triangle are inscribed in the same circle. Find the ratio between (1) their sides, and (2) their areas.

2. Compare the areas of regular pentagons inscribed within, and described about, a given circle.

3. Find the value of the interior angle of any regular polygon of n sides.

4. Show that the square described about a circle is equal to $\frac{4}{3}$ of the inscribed duodecagon.

5. Show that the area of a regular polygon inscribed in a circle is a mean proportional between the areas of an inscribed and circumscribed polygon of half the number of sides.

6. Find the radii of the inscribed, and each of the escribed circles of the triangle ABC, when $a = 5$, $b = 7$, $c = 9$.

7. With the usual notation, prove the following relations :—

(1) $R = \frac{abc}{4S}$.

(2) $\sqrt{r \cdot r_1 \cdot r_2 r_3} = S$.

(3) $r_1 + r_2 + r_3 - r = 4R$.

(4) $\frac{1}{r_1 - r} + \frac{1}{r_2 + r_3} = \frac{4R}{a^2}$.

(5) $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{S^2}$.

(6) $R (\sin. A + \sin. B + \sin. C) = s$.

(7) $4 R^2 (\cos. A + \cos. B \cos. C) = bc$.

8. Show that the distance between the centre of the inscribed circle, and that of the escribed circle touching the side BC, is

$$a \sec. \frac{A}{2}.$$

9. If C is a right angle, show that $R + r = \frac{a + b}{2}$.

10. If A is a right angle, show that $r_2 + r_3 = a$.

11. If O is the centre of the circle described about the triangle ABC, and AO is produced to meet BC at D, show that

$$DO \cos. (B - C) = AO \cos. A.$$

12. Find the angle of the sector in which the chord of the arc is three times the radius of the circle inscribed in the sector.

13. In any triangle show that the area of the inscribed circle is to the area of the triangle as π to $\cot. \frac{A}{2} \cot. \frac{B}{2} \cot. \frac{C}{2}$.

14. Show that in any triangle the radius of the inscribed

circle bears to the radius of the circumscribed circle the ratio of $4 \sin.\frac{A}{2} \sin.\frac{B}{2} \sin.\frac{C}{2} : 1$.

15. If DEF be the triangle formed by the lines joining the feet of the perpendiculars from A, B, C upon the opposite sides, then the radius of the circle inscribed within DEF is to the radius of the circle inscribed within ABC as

$$\text{Cos.}A \cos.B \cos.C : 2 \sin.\frac{1}{2}A \sin.\frac{1}{2}B \sin.\frac{1}{2}C.$$

FORMULÆ OF REFERENCE (II.).

1. In any plane triangle, ABC,

(1) $\frac{\text{Sin.}A}{a} = \frac{\sin.B}{b} = \frac{\sin.C}{c}$ (art. 133)

(2) $\text{Cos.}C = \frac{a^2+b^2-c^2}{2ab}$ (art. 134)

(3) $a = c\cos.B + b\cos.C$ (art. 135)

(4) $\text{Sin.}\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \cos.\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (art. 137)

(5) $\text{Hav.}A = \frac{(s-b)(s-c)}{bc}$ (art. 138)

(6) $\text{Tan.}\frac{A-B}{2} = \frac{a-b}{a+b}\cot.\frac{C}{2}$ (art. 140)

(7) $S = \frac{1}{2}bc\sin.A$ (art. 141)

(8) $S = \sqrt{s(s-a)(s-b)(s-c)}$ (art. 142)

(9) $r = \frac{S}{s}$ (art. 170)

(10) $r_1 = \frac{S}{s-a}, r_2 = \frac{S}{s-b}, r_3 = \frac{S}{s-c}$ (art. 172)

(11) $R = \frac{a}{2\sin.A} = \frac{b}{2\sin.B} = \frac{c}{2\sin.C}$ (art. 173)

2. The area of a circle of radius $r = \pi r^2$ (art. 179)

ANSWERS TO THE EXAMPLES.

I. (page 4).

1. 44. 2. 60 yards. 3. 40.
4. $\frac{3a}{b}$. 5. 1ft. 9in. 6. $2\frac{1}{3}$ inches.

II. (page 5).

1. 11ft. 8in. 2. As 2 to 1. 3. 48 feet. 4. 8·6 feet nearly.
5. 24 feet. 6. 159·8 yards nearly. 7. $\frac{2}{\sqrt{3}}a$.
8. As 1 to $\sqrt{2}$. 9. 10 feet. 10. 23·6 inches nearly.

III. (page 9).

1. 3·93 inches. 2. $45\frac{9}{11}$ inches. 3. 15min. 54·5 sec.
4. 5·971 inches. 5. ·278 inch. 6. 158·7 yards.

IV. (page 12).

1. (1) $19° 5' 55''$. (2) $100° 16' 3''$. (3) $39° 23' 27''$.
(4) $60°$. (5) $22° 30'$. (6) $191° 15'$.

2. (1) $\frac{\pi}{6}$. (2) $\frac{5}{12}\pi$. (3) $\frac{7}{3}\pi$. (4) $\frac{\pi}{3}$.

3. $6\frac{2}{3}$. 4. $7° 30'$. 5. $\frac{4\pi}{9}, \frac{4\pi}{9}, \frac{\pi}{9}$. 6. $\frac{2}{\pi}$.

8. 2·356 nearly. 9. ·856 nearly. 10. $30°, 50°, 100°$.
11. 1·494 nearly. 12. $\frac{2\pi}{3}$. 13. $108°$, or $\frac{3\pi}{5}$.
14. $5\frac{8}{11}$ feet. 15. $1°·018$ nearly. 16. 2,269 miles nearly.

VI. (page 27).

1. (1) $62° 22' 12''$. (2) $39° 43' 22''$. (3) $-15°$.
(4) $127°$. (5) $\frac{\pi}{6}$. (6) $\frac{7\pi}{6}$.

2. (1) $100° 23' 45''$. (2) $78° 40' 17''$. (3) $-20°$.
(4) $250°$. (5) $\frac{3\pi}{5}$. (6) $\frac{8\pi}{7}$.

3. (1) Sin. $x =$ cos. A cosec. A.
(2) Cos. $x =$ cos. A cosec. A.
(3) Sin. $x = -$ sin. A cos. A sec.^{2} B.

VIII. (page 34).

1. $\text{Sin. } \theta = \dfrac{1}{\sqrt{1 + \cot.^2 \theta}}$.

2. $\text{Sin. } \theta = \dfrac{\sqrt{\sec.^2 \theta - 1}}{\sec. \theta}$, $\cot. \theta = \dfrac{1}{\sqrt{\sec.^2 \theta - 1}}$.

3. $\text{Sin. } \theta = \sqrt{2 \text{ vers. } \theta - \text{vers.}^2 \theta}$, $\cos. \theta = 1 - \text{vers. } \theta$, &c.

4. $\text{Tan. A} = \dfrac{3}{4}$.

5. $\text{Cot. A} = \dfrac{1}{\sqrt{3}}$.

6. $\text{Sin. } \theta = \dfrac{2}{3}$, $\sec. \theta = \dfrac{3}{\sqrt{5}}$.

7. $\text{Cot. A} = \dfrac{5}{2\sqrt{6}}$.

8. $\text{Tan. A} = \dfrac{1}{3\sqrt{7}}$.

9. $\text{Cosec. A} = \dfrac{\sqrt{a^2 + b^2}}{a}$.

10. $a^2b^2 + a^2 = 1$.

IX. (page 38).

1. (1) 1. (2) $\frac{1}{2}$. (3) $\sqrt{3}$. (4) -1.
   (5) -2. (6) $\sqrt{3}$. (7) -2. (8) $\dfrac{1}{2}$.
   (9) $-\dfrac{1}{2}$ (10) $\sqrt{3}$. (11) 0. (12) 0.

2. (1) $n\pi + \dfrac{\pi}{4}$. (2) $n\pi + (-1)^n \dfrac{\pi}{6}$. (3) $2n\pi \pm \dfrac{2\pi}{3}$.
   (4) $(2n+1)\pi$. (5) $n\pi \pm \dfrac{\pi}{4}$. (6) $n\pi \pm \dfrac{\pi}{3}$.

X. (page 43).

17. $\dfrac{220}{221}$. 18. 3. 19. $-\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $2+\sqrt{3}$, $2-\sqrt{3}$.

XIII. (page 47).

1. Cos. (A + B) + cos. (A − B).
2. Sin. 5A + sin. 3A.
3. Cos. A − cos. 7A.
4. Sin. 7A − sin. 2A.
5. Sin. (2A + B) − sin. B.
6. $\text{Cos.}\left(\text{A} + \dfrac{3\text{B}}{2}\right) + \cos. \dfrac{3\text{B}}{2}$.
7. Sin. 70° + sin. 30°.
8. Cos. 15° − cos. 35°.

XVI. (page 56).

1. (1) + +. (2) + −. (3) + −. (4) − −.
   (5) + +. (6) + −.

4. Between 270° and 450°.

5. (1) Sin. 72° = cos. 18°. (2) Cos. 54° = sin. 36°.
   (3) $\text{Sin.} 9° = \dfrac{1}{4}\left(\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right)$.
   (4) $\text{Cos.} 9° = \dfrac{1}{4}\left(\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}\right)$.
   (5) Cos. 81° = sin. 9°.

XVII. (page 58).

1. $\theta = \frac{\pi}{2}$. 2. $\theta = \frac{\pi}{6}$. 3. $\theta = \frac{3\pi}{4}$. 4. $\theta = \frac{\pi}{4}$.

5. $\theta = \frac{\pi}{6}$. 6. $\theta = \frac{\pi}{3}$. 7. $\theta = \frac{\pi}{4}$. 8. $\theta = \frac{\pi}{4}$.

9. $\theta = \frac{\pi}{6}$. 10. $\theta = \frac{\pi}{3}$. 11. $\theta = \frac{\pi}{6}$. 12. $\theta = \frac{\pi}{4}$.

13. $\theta = \frac{\pi}{4}$. 14. $\theta = \frac{\pi}{6}$. 15. $\theta = \frac{5\pi}{9}$. 16. $\theta = \frac{\pi}{4}$ or $\frac{\pi}{6}$.

17. $\theta = \frac{\pi}{12}$, or 0.

XVIII. (page 59).

1. $\theta = 2n\pi + \frac{\pi}{2}$.
2. $\theta = n\pi + \frac{\pi}{4}$.
3. $\theta = 2n\pi \pm \frac{\pi}{4}$.
4. $\theta = n\pi \pm \frac{\pi}{6}$.
5. $\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}$.
6. $\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{6}$.
7. $\theta + \frac{\pi}{4} = n\pi \pm \frac{\pi}{3}$.
8. $\theta = 2n\pi \pm \frac{3\pi}{4}$.
9. $\frac{3\theta}{4} = n\pi + \frac{\pi}{2}$ or $\frac{\theta}{4} = 2n\pi \pm \frac{\pi}{3}$.
10. $2\theta = n\pi + \frac{\pi}{2}$ or $\theta = 2n\pi \pm \frac{2\pi}{3}$.

XIX. (page 60).

15. $x = 2$. 16. $x = 2$. 17. $x = 3$.

18. $x = \sqrt{\frac{2}{a}}$. 19. $x = \frac{1}{2}$. 20. $x = \frac{1}{2\sqrt{3}}$.

XX. (page 67).

1. 4·477121, $\bar{4}$·301030, ·352182, $\bar{2}$·647818.
2. $\bar{1}$·741167.
3. $\bar{2}$·778074.
4. 1·732395, ·94639.
5. (a) -5, (b) $-\frac{1}{6}$, (c) $\frac{26}{9}$.
6. $\bar{2}$·698970, ·150515, 6·64386.
7. $-$ ·6751.
8. $x = 3$·1547.
9. ·5172818.
10. $\frac{4}{3}$, $\frac{18}{5}$.
11. $x = 75$.
12. ·301030.
13. $\frac{mc + m}{c}$.
17. $x = 5$·435.
18. 486.
19. By dividing each logarithm by 3.

XXI (page 81).

1. L sin. 45° = 9·849485, L tan. 45° = 10·000000, L sin. 60° = 9·937531, L cos. 60° = 9·698970, L cot. 30° = 10·238561, L tan. 210° = 9·761439.

3. 9·763188.
4. 9·937238.
5. 10·035974.
6. 9·984944.
7. 9·778151.
8. 700000.
9. 9·923592.
10. 9·875061.
11. 200000.
12. $\frac{1}{2}$ and $\frac{1}{5}$.

XXIV. (page 114).

1. The sides as $\sqrt{2}$ to $\sqrt{3}$; the areas as 8 to $3\sqrt{3}$.
2. As $3 + \sqrt{5}$ to 8.
3. $\pi\left(1 - \frac{2}{n}\right)$.
6. $r = \frac{1}{2}\sqrt{11}$, $r_1 = \frac{21}{22}\sqrt{11}$, $r_2 = \frac{3}{2}\sqrt{11}$, $r_3 = \frac{7}{2}\sqrt{11}$.
12. 60°.

Part II.

SPHERICAL TRIGONOMETRY

CHAPTER I.

THE GEOMETRY OF THE SPHERE.

1. A SPHERE is a solid bounded by a surface every point of which is equally distant from a certain fixed point, which is called the *centre* of the sphere.

The straight line which joins any point of the surface with the centre is called a *radius*.

A straight line drawn through the centre, and terminated both ways by the surface, is called a *diameter*.

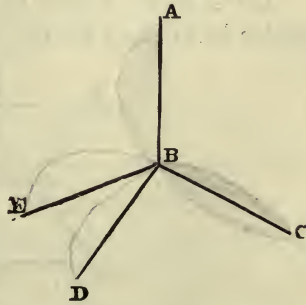
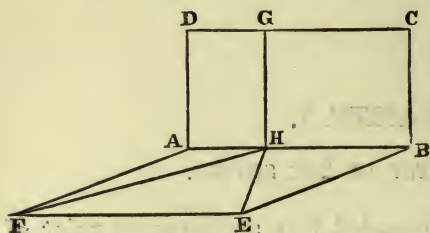
2. In order to establish the fundamental theorems upon which the science of spherical trigonometry is based, it will be necessary to take for granted certain definitions and propositions of the eleventh book of Euclid. It will be convenient to give here the definitions which will be required, and the enunciations of the propositions, the truth of which we shall have afterwards to assume.

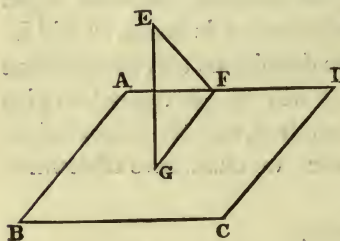
DEFINITIONS.

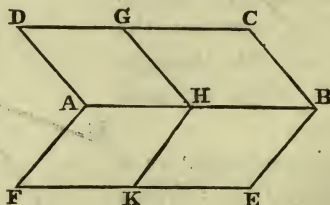
3. *A straight line is perpendicular, or at right angles, to a plane when it makes right angles with every straight line meeting it in that plane* (Euc. XI. *def.* 3).

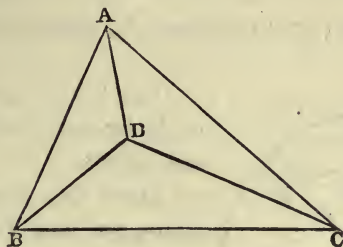
A
B
E
C
D

Thus if AB be at right angles to the plane which contains the straight lines BC, BD, BE, then the angles ABC, ABD, ABE are each of them a right angle.

A plane is perpendicular to a plane, when the straight lines drawn in one of the planes, perpendicular to the common section of the two planes, are perpendicular to the other plane (Euc. XI. *def.* 4).

Thus if the plane ABCD be at right angles to the plane ABEF, and GH be drawn in the plane ABCD at right angles to AB, the common section, the angles which GH makes with HE, HF, &c., any straight lines drawn in the plane ABEF, are all right angles.

The inclination of a straight line to a plane is the acute angle contained by that straight line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane meets the same plane (Euc. XI. *def.* 5).

Thus the inclination of the straight line EF to the plane ABCD is found by dropping EG perpendicular to the plane from any point E in EF, and joining FG. The angle EFG is the inclination of EF to the plane ABCD.

The inclination of a plane to a plane is the acute angle contained by two straight lines drawn from any the same point of their common section, at right angles to it, one in one plane, and the other in the other plane (Euc. XI. *def.* 6).

Let ABCD, ABEF be two planes, of which the common section is AB, and let HG, HK be drawn in the two planes at

right angles to AB, then the angle GHK measures the inclination of the two planes.

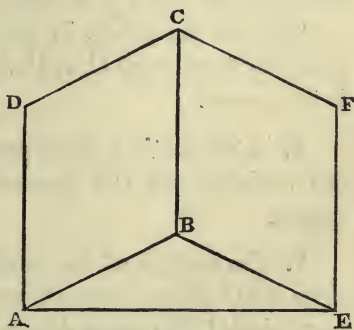
A solid angle is that which is made by more than two plane angles, which are not in the same plane, meeting at one point (Euc. XI. *def.* 9).

Thus in the figure the three plane angles ADB, BDC, CDA contain a solid angle at D.

Theorems.

4. *If a straight line stand at right angles to each of two straight lines at the point of their intersection, it shall also be at right angles to the plane which passes through them; that is, to the plane in which they are* (Euc. XI. 4).

In the figure attached in the preceding article to def. 3, if ABD, ABE be both right angles, AB will be at right angles to the plane containing BD, BE.

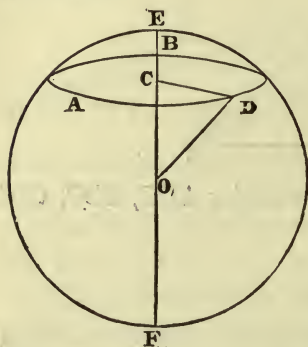
If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane (Euc. XI. 19).

If the two planes ABCD, EBCF be each of them perpendicular to the plane ABE, then BC, the common section of the first two planes, is perpendicular to the plane ABE.

If a solid angle be contained by three plane angles, any two of them are together greater than the third (Euc. XI. 20).

Every solid angle is contained by plane angles, which are together less than four right angles (Euc. XI. 21).

Thus in the figure attached to def. 9, any two of the three

plane angles ADB, BDC, CDA are greater than the third angle, and the sum of the three angles is together less than four right angles.

5. *The section of the surface of a sphere, made by any plane, is a circle.*

Let AB be the section of a sphere made by any plane, O the centre of the sphere.

Draw OC perpendicular to the plane, so that OC produced both ways is a diameter of the sphere.

Take any point D in the section, and join OD, DC.

Since OC is perpendicular to the plane, the angle OCD is a right angle (Euc. XI. *def.* 3).

Therefore $CD^2 = OD^2 - OC^2$.

But O and C being fixed points, OC is constant, or has a fixed value.

And OD is constant, for it is a radius of the sphere.

Therefore CD also is constant; that is, it has the same value wherever the point D is taken in the section.

Thus all the points in the plane section are equally distant from the fixed point C, so that the section is a circle, of which C is the centre.

6. If the cutting plane pass through the centre of the sphere OC vanishes, and CD becomes equal to OD, the radius of the sphere.

7. The section of the surface of a sphere is called a *great circle* if the plane passes through the centre of the sphere, and a *small circle* if the plane does not pass through the centre of the sphere. The radius of a great circle is therefore equal to the radius of the sphere.

8. Through the centre of a sphere and any two points on its surface only one plane can in general be drawn; but if the two points are situated at the extremities of a diameter, an infinite number of such planes can be drawn.

If therefore two points are not at the extremity of a diameter, only one great circle can be drawn through them, which is unequally divided at the two points. The shorter of the two arcs is commonly spoken of as *the* arc of a great circle joining the two points.

9. The *axis* of any circle of a sphere is that diameter of the sphere which is perpendicular to the plane of the circle, and the extremities of the axis are known as the *poles* of the circle.

Thus in the figure to art. 5, EF is the axis of the circle ABD, and E, F are the poles of the circle.

In the case of a great circle the poles are equally distant from the plane of the circle.

With a small circle this is not so, and the two poles are called respectively the *nearer* and the *further* pole. Thus in the figure, E is the *nearer* pole and F the *further* pole, of which the nearer is commonly known as *the* pole.

10. *A pole of a circle is equally distant from any point in the circumference.*

Let O be the centre of the sphere, and AB be any circle of the sphere. Let PP′ be the axis of the circle. Then C, the point in which the axis meets the plane of the circle, is the centre of the circle (art. 5), P,P′ being the poles.

Take any point D in the circumference of the circle, and join CD, PD.

Then $PD^2 = PC^2 + CD^2$.

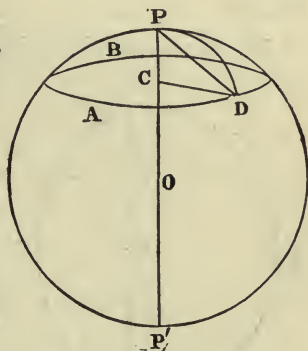
But PC, CD are the same in length wherever D may be.

Therefore PD is constant.

Hence the length of the straight line joining the pole P with any point in the circumference of the circle is invariable.

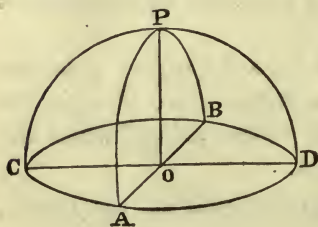
Similarly it may be shown that the pole P′ is equally distant from every point in the circumference of the circle.

Again, let a great circle be described through the points P and D, when D is any point in the circle AB.

Since the length of the chord PD is the same for all points

in the circumference AB, and since in equal circles the arcs which are cut off by equal straight lines are equal (Euc. III. 28).

Therefore the arc of a great circle intercepted between P and D is constant for all positions of D on the circle AB.

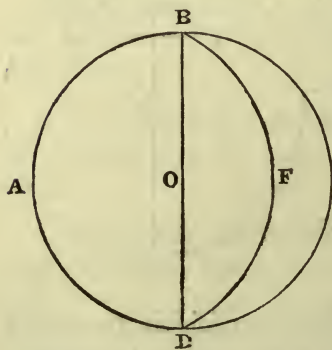
Thus the distance of a pole of a circle from every point of the circle is constant, whether the distance be measured by the straight line joining the points, or by the arc of a great circle intercepted between the points.

11. *The arc of a great circle which is drawn from a pole of a great circle to any point in its circumference is a quadrant.*

Let ACBD be a great circle, and P its pole, O being the centre of the sphere.

Join OP, and draw any diameters of the circle AOB, COD.

Because OP is at right angles to the plane of the circle, therefore POA, POC, POB, POD are all right angles (Euc. XI. *def.* 3).

Therefore the arcs PA, PC, PB, PD are all quadrants (Euc. VI. 33).

12. *Two great circles bisect each other.*

Let BAD, BFD be two arcs of great circles, intersecting one another in the points B, D.

Then BD, the common section of the planes of the two great circles, is a straight line (Euc. XI. 3).

Join BD.

And because the centre of the sphere lies in each of the planes of the circles, it must therefore be a point in their common section, as O.

Therefore BD must be a diameter of the sphere, and of each of the two circles BAD, BFD.

Therefore the two circles BAD, BFD bisect each other.

13. *If the arcs of great circles joining a point on the surface of*

a sphere with two other points on the surface of the sphere, which are not at opposite extremities of the same diameter, be each of them quadrants, then the first point is a pole of the great circle through the last two points.

Let O be the centre of the sphere, P a point on the surface, and PA, PB be two arcs, each equal to a quadrant.

Through the points A, B let a great circle be described, and join OA, OB, OP.

Then, because PA, PB are quadrants, therefore POA, POB are each right angles.

Therefore PO stands at right angles to each of the straight lines OA, OB at the point of their intersection.

Therefore PO is at right angles to the plane AOB in which they are (Euc. XI. 4).

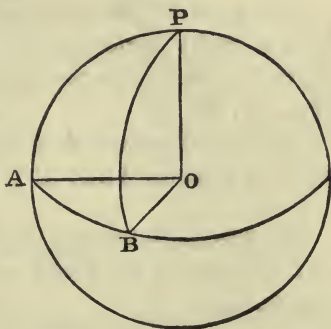
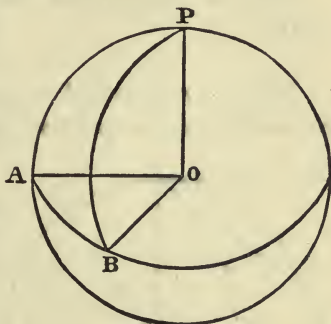
Therefore OP is a portion of the axis of the circle AB (art. 9), and P, being an extremity of the axis, is a pole of the circle AB.

14. *If from a point on the surface of a sphere there can be drawn two arcs of great circles, not parts of the same great circle, the planes of which are at right angles to the plane of a given circle, that point is a pole of the given circle.*

Let O be the centre of the sphere, AB a great circle of the sphere, and PA, PB two great circles whose planes are at right angles to the plane of AB.

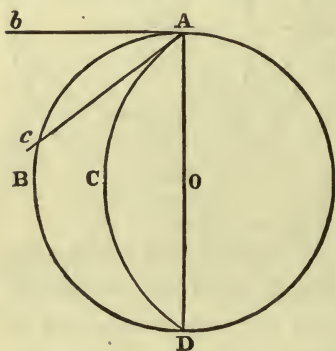
Then, since the planes of the circles PA, PB are at right angles to the circle AB, therefore PO, their common section, is at right angles to AB (Euc. XI. 19).

Therefore P is a pole of the great circle AB.

Similarly P will be the pole of any small circle which has its plane parallel to that of AB, so that the planes of PA, PB are at right angles to it.

Cor. Hence it follows that if in the circumference of a given circle any two points be taken, and arcs drawn through the two points at right angles to the given circle, the point of intersection of these two arcs is a pole of the given circle.

15. Great circles which pass through the poles of another great circle are said to be *secondaries* to that circle.

Thus, in the figure of art. 14, PA, PB are secondary circles to AB.

16. *Def.* A spherical angle is the inclination of two arcs of great circles at the point where they meet.

Thus let ABD, ACB be two great circles intersecting in A, D, and having the straight line AD for their common section.

And let A*b* be the tangent at the point A to the circle ABD, and A*c* the tangent at the point A to the circle ACD.

Then BAC is a spherical angle.

And since at the point A the two arcs AB, AC have the same directions as their tangents, the spherical angle BAC is measured by the angle *b*A*c*.

But since A*b*, A*c* are straight lines drawn in the planes ABD, ACD at right angles to AD, the common section of the two planes, the angle *b*A*c* measures the inclination of the two planes (Euc. XI. *def.* 6).

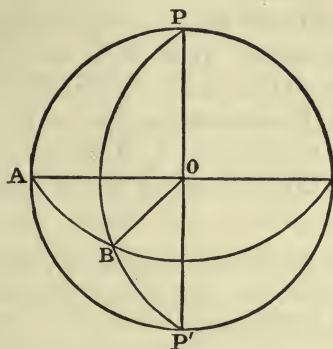
Hence the spherical angle BAC is equal to the inclination of the two planes ABD, ACD.

17. *The angle between any two great circles is measured by the arc intercepted by the circles on the great circle to which they are secondaries.*

Let AB be the arc of a great circle of which P is the pole, so that PA, PB are arcs of great circles, secondaries to AB, and let O be the centre of the sphere.

Then the spherical angle APB will be measured by the arc AB.

Because P is the pole of the circle AB,

Therefore the angles POA, POB are each of them right angles.

Therefore the angle AOB represents the inclination of the planes of the circles PA, PB (Euc. XI. *def.* 6).

But the angle AOB is measured by the arc AB.

Therefore the arc AB measures the inclinations of the planes of the circles PA, PB.

18. *The angle subtended at the centre of a sphere by the arc of a great circle which joins the poles of two great circles is equal to the inclinations of the planes of the great circles.*

Let O be the centre of the sphere, CD, CE the two great circles intersecting in the point C, A and B the poles of CD and CE respectively.

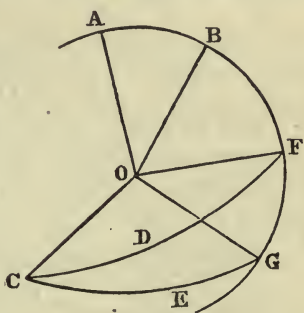
Let a great circle be drawn through A, B, meeting CD, CE in F, G respectively.

Join OA, OB, OC, OF, OG.

Because A is the pole of the circle CDF, therefore the angle AOC is a right angle.

And because B is the pole of the circle CEG, therefore the angle BOC is a right angle.

Therefore OC stands at right angles to the two straight lines OA, OB at the point of their intersection.

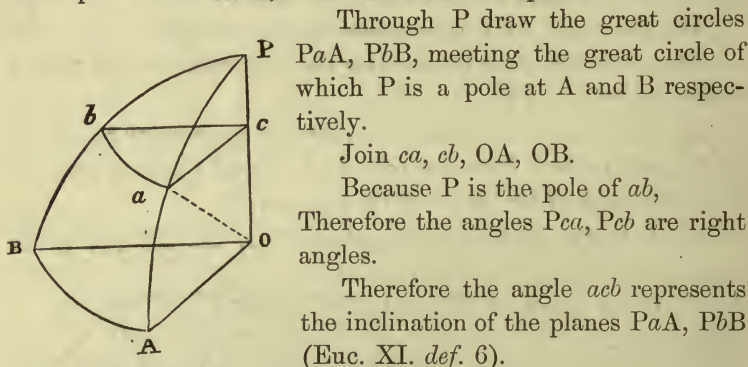
Therefore it is at right angles to the plane AOB (Euc. XI. 4).

Therefore OC is at right angles to OF, OG, two straight lines in the plane AOB.

Hence, FOG represents the inclination of the two planes OCD, OCE (Euc. XI. *def.* 6).

And since the angles AOF, BOG are both right angles, therefore the angle AOB = AOF − BOF = BOG − BOF = FOG.

That is, the angle subtended at the centre of the sphere by the arc of a great circle joining A, B, the poles of the two circles CD, CE, is equal to the inclination of the planes of the circles CD, CE.

19. *To compare the arc of a small circle, subtending any angle at the centre of that circle, with the arc of a great circle subtending the same angle at its centre.*

Let ab be the arc of a small circle, c the centre of the circle, P the pole of the circle, O the centre of the sphere.

Through P draw the great circles PaA, PbB, meeting the great circle of which P is a pole at A and B respectively.

Join ca, cb, OA, OB.

Because P is the pole of ab,

Therefore the angles Pca, Pcb are right angles.

Therefore the angle acb represents the inclination of the planes PaA, PbB (Euc. XI. *def.* 6).

And because P is the pole of AB;

therefore the angles POA, POB are right angles.

Therefore the angle AOB represents the inclination of the planes PaA, PbB (Euc. XI. *def.* 6).

Therefore the angle acb is equal to the angle AOB.

And $\frac{\text{arc } ab}{\text{radius } ca} = \frac{\text{arc AB}}{\text{radius OA}}$, for these fractions represent the circular measure of the angle acb or AOB (Part I. art. 17).

Therefore $$\frac{\text{arc } ab}{\text{arc AB}} = \frac{\text{radius } ca}{\text{radius OA}}.$$

Hence $$\frac{\text{arc } ab}{\text{arc AB}} = \frac{ca}{\text{OA}} = \frac{ca}{\text{O}a} = \sin.\ \text{PO}a = \cos.\ a\text{OA} = \cos.\ a\text{A}.$$

Therefore the length of the arc ab is equal to the length of the arc AB multiplied by the cosine of the arc aA.

This result is of very great importance, and is of frequent use in the problems of navigation and nautical astronomy.

CHAPTER II.

ON CERTAIN PROPERTIES OF SPHERICAL TRIANGLES.

20. A SPHERICAL triangle is the figure formed on the surface of a sphere by three arcs of great circles intersecting one another.

Thus in the figure, if O be the centre of the sphere, the great circles ABG, ACG, CBD intersecting at A, B and C will form on the surface of the sphere a figure ABC, which is called a spherical triangle.

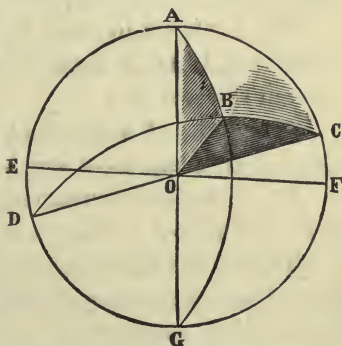
The arcs AB, BC, CA are the sides of the triangle, and the angles formed by the arcs at the points A, B, C are called the angles of the triangle.

And, as in plane trigonometry, it is usual to denote the angles by the letters A, B, C, and the sides respectively opposite to them by the letters a, b, c.

21. The sides of a spherical triangle are therefore arcs of great circles, and are proportional to the three angles which at the centre of the sphere form the solid angle O.

In the various relations which will hereafter be established between the six parts of a spherical triangle, we shall in general only have to consider the angles subtended at the centre of the sphere by the arcs of great circles which form the sides of the triangle, and not the actual lengths of the sides themselves; and in the several formulæ functions of the angles so subtended, such as the sine and cosine of these angles, will appear.

Thus by cos. a is to be understood the cosine of the angle subtended at the centre by the arc BC, or the angle BOC, in the figure of art. 20.

It may however be observed that if the radius of a particular

sphere be given we may always obtain the actual length of an arc which subtends a given angle at its centre, by means of the circular measure of the angle.

Thus, let us suppose that the sphere depicted in art. 20 represents the earth, and that B, C are two places on the earth's surface. Let us also suppose that the arc BC joining the two places subtends an angle of 45°, and that the radius of the earth is known to be 4,000 miles. Then, by Part I. art. 17, the circular measure of the angle BOC is $\frac{BC}{OB}$.

Therefore $$\frac{BC}{OB} = \frac{BC}{4000} = \frac{\pi}{4};$$

and $$BC = \frac{\pi}{4}\,4000, \text{ or } 3146 \text{ miles nearly.}$$

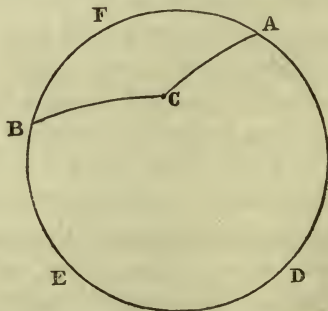
22. As shown in art. 16, the spherical angle formed at A by the arcs AB, AC is equal to the inclination of the planes which contain the two arcs.

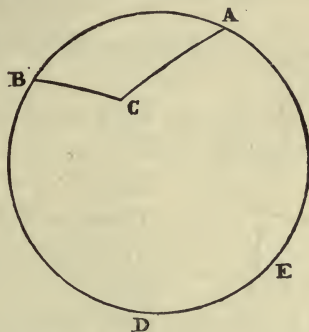
The angles of a spherical triangle are therefore the inclinations of the plane faces which form the solid angle.

23. It is found convenient in spherical trigonometry to agree that each side of a triangle shall be less than a semicircle.

Thus in the figure let ADEF be a great circle, divided by A and B into two unequal arcs ADEB, AFB, of which ADEB is greater than a semicircle.

Thus the triangle ABC may be regarded as formed by the arcs CA, CB, AFB, or by the arcs CA, CB, ADEB.

It is to be understood, when the triangle ABC is spoken of,

that it is the first of these, namely, the triangle formed by the arcs CA, CB, AFB which is meant.

24. And if each side of a spherical triangle is less than a semicircle, it will follow also that *each angle of a spherical triangle will be less than two right angles.*

For let a triangle be formed by BC, CA, and BDEA, having the angle BCA greater than two right angles.

Let E be the point where BC if produced will meet AE.

Then BDE is a semicircle (art. 12).

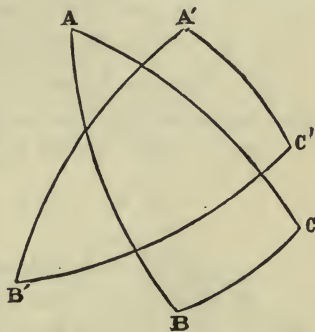
Therefore BDEA is greater than a semicircle; that is, the proposed triangle is one which we have agreed to exclude from consideration.

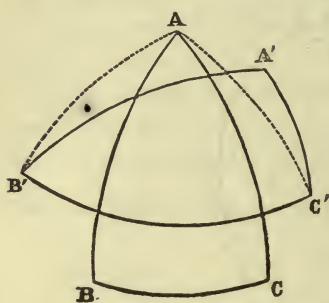
25. *Polar triangle.*

Let ABC be a spherical triangle, and let the points A′, B′, C′ be those poles of BC, CA, AB which lie on the same sides of those arcs as the opposite angles A, B, C.

Then the triangle A′B′C′ is said to be the *polar triangle* of the triangle ABC.

Since each side of a spherical triangle has two poles, *eight* triangles in all may be formed having for their angular points poles of the sides of the given triangle, but in only one of these do the poles A′, B′, C′ lie *towards the same parts* with the corresponding angles A, B, C, and this is the one which is known as the polar triangle.

The triangle ABC is known as the *primitive* triangle with respect to the triangle A′B′C′.

If one triangle be the polar triangle of another, the latter will be the polar triangle of the former.

Let ABC be a triangle, and A′B′C′ its polar triangle.

It is required to show that the triangle ABC is the polar triangle of A′B′C′.

First to prove that A is a pole of B′C′.

Since B′ is a pole of AC,
Therefore B′A is a quadrant.

And since C′ is a pole of AB,
Therefore C′A is a quadrant.

Hence since AB′, AC′ are both quadrants, A is a pole of B′C′ (art. 13).

Next to show that A, A′ lie upon the same side of B′C′.

Since A′ is a pole of BC, and A, A′ lie upon the same side of BC,

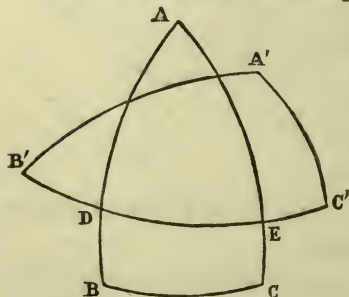
Therefore A′A is less than a quadrant.

And since the arc AA′, drawn from A, a pole of B′C′, to A′, is less than a quadrant, the two points A, A′ must lie upon the same side of B′C′.

Similarly it may be shown that B is a pole of A′C′, and that B, B′ are upon the same side of A′C′. Also that C is a pole of A′B′, and that C and C′ are on the same side of A′B′.

Thus ABC is the polar triangle of A′B′C′.

26. *The sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle.*

Let the side B′C′ of the polar triangle, produced if necessary, meet the sides AB, AC, produced if necessary, in D, E.

Then since A is the pole of B′C′, the arc DE measures the angle A (art. 17).

Because B′ is a pole of AC,
Therefore B′E is a quadrant.
And because C′ is a pole of AB,
Therefore C′D is a quadrant.

Therefore B′E, C′D are together equal to a semicircle.

But the arcs B′E, C′D together make up the arcs B′C′ and DE, that is, the arc B′C′ and the angle A.

Therefore the side B′C′ and the angle A are supplementary to each other.

Similarly it may be shown that A′C′, A′B′ are supplementary respectively to B, C.

And since ABC is the polar triangle of A′B′C′, it follows that BC, CA, AB are respectively the supplements of the angles A′, B′, C′.

27. From these properties a primitive triangle and its polar triangle are sometimes spoken of as *supplemental triangles*.

Thus if A, B, C, a, b, c denote the angles and the sides of the primitive triangle, and A′, B′, C′, a', b', c' those of the polar triangle, all expressed in circular measure, we have

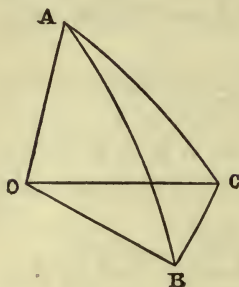
$$A' = \pi - a,\ B' = \pi - b,\ C' = \pi - c$$
$$A = \pi - a',\ B = \pi - b',\ C = \pi - c'.$$

28. *If a general equation be established between the sides and the angles of a spherical triangle, the supplements of the sides and angles respectively may be substituted for the angles and sides which are involved in the equation.*

For since the equation is general it is true for any triangle, and it holds, therefore, when a', b', c', A′, B′, C′ (the sides and angles of the polar triangle), are substituted for a, b, c, A, B, C respectively. In the equation as it then stands we may substitute for the sides and angles of the polar triangle their equivalents drawn from the primitive triangle; thus,

$$a' = \pi - A,\ b' = \pi - B,\ \&c..$$

If, then, a general theorem be established connecting the sides and angles of a spherical triangle, we may substitute for each side the supplement of the corresponding angle, and for each angle the supplement of the corresponding side, and the relation between the sides and angles so obtained will be also true for all triangles. The great importance of this result will appear later.

29. *Any two sides of a spherical triangle are together greater than the third.*

For any two of the three plane angles AOB, AOC, BOC, which form the solid angle at O, are together greater than the third (Euc. XI. 20).

Therefore any two of the arcs AB, AC, BC, which measure these angles respectively, are together greater than the third.

From this proposition it is clear that any side of a spherical triangle is greater than the difference of the other two.

30. *The sum of the three sides of a spherical triangle is less than the circumference of a great circle.*

The sum of the three plane angles which form the solid angle at O is less than four right angles (Euc. XI. 21).

Therefore the sum of the circular measures of these angles is less than the circular measure of four right angles.

Therefore $\frac{AB}{OA} + \frac{BC}{OA} + \frac{AC}{OA}$ is less than 2π.

Therefore $AB + BC + AC$ is less than $2\pi \times OA$; that is, the sum of the three arcs is less than the circumference of a great circle.

31. *The three angles of a spherical triangle are together greater than two and less than six right angles.*

Let A, B, C be the angles of a spherical triangle, and let a', b', c' be the sides of the polar triangle.

Then by art. 30, $a' + b' + c'$ is less than 2π.

Therefore $\pi - A + \pi - B + \pi - C$ is less than 2π;

Or, $\pi - (A + B + C)$ is less than O.

That is, $A + B + C$ is greater than π.

And since each of the angles A, B, C is less than π, the sum of A, B, C is less than 3π.

32. *The angles at the base of an isosceles triangle are equal to one another.*

Let ABC be a spherical triangle having the side AC equal to the side BC, and let O be the centre of the sphere.

At the point A draw a tangent to the arc AC, and let this meet OC produced in D.

Join BD.

Then in the triangles AOD, BOD, the side AO is equal to BO, OD is common, and the angle AOD is equal to the angle BOD, since these two angles are measured respectively by the equal arcs AC, CB. Therefore BD = AD, and OBD = the right angle OAD.

Hence BD touches the arc BC.

At the points A, B draw tangents AT, BT to the arc AB, and join TD.

Then in the two triangles ATD, BTD we have AT = BT, being tangents drawn to a circle from the same point T, and AD is equal to BD, and TD common to both triangles.

Therefore the angle TAD is equal to the angle TBD.

And TAD measures the angle BAC, and TBD measures ABC (art. 16).

33. *If two angles of a spherical triangle are equal, the opposite sides are equal.*

In the spherical triangle ABC let the angles A, B be equal.

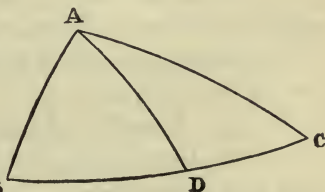
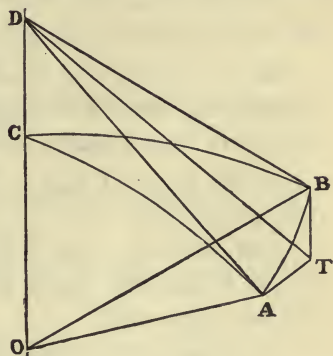
Therefore in the polar triangle the sides a', b' are equal. Hence in the same triangle the angles A′, B′ are equal (art. 32).

Therefore in the primitive triangle the sides a, b are equal.

34. *If one angle of a spherical triangle be greater than another, the side opposite to the greater angle is greater than the side opposite to the less angle.*

Let ABC be a spherical triangle, and let the angle A be greater than the angle B.

At A make the angle BAD equal to the angle B.

Then because the angle BAD is equal to the angle ABD, therefore the arc DB is equal to the arc DA (art. 33).

But the two arcs DA, DC are together greater than AC.

Therefore the two arcs DB, DC are together greater than the arc AC; that is, the side BC is greater than AC.

35. *If one side of a spherical triangle be greater than another, the angle opposite to the greater side is greater than the angle opposite to the less side.*

In the spherical triangle ABC let the side BC be greater than the side AC.

Then shall the angle BAC be greater than the angle ABC.

For if not, the angle BAC must be either equal to or less than the angle ABC.

If the angle BAC were equal to the angle ABC, then would the side BC be equal to the side AC (art. 33).

But by hypothesis BC is not equal to AC.

Similarly if the angle BAC were less than the angle ABC, the side BC would be less than the side AC (art. 34).

But by hypothesis BC is not less than AC.

Therefore the angle BAC must be greater than the angle ABC.

From these two propositions it follows that in any spherical triangle $A-B$, $a-b$ must have the same sign.

36. The following is a summary of the more important properties of spherical triangles established in this chapter.

1. Each side of a spherical triangle must be less than a semicircle (art. 23).

2. Each angle must be less than two right angles (art. 24).

3. Any two sides of a spherical triangle are together greater than the third (art. 29).

4. If two sides of a spherical triangle are equal, the angles opposite to them are equal, and conversely (arts. 32, 33).

5. The greater side is opposite to the greater angle, and conversely (arts. 34, 35).

6. $A - B$ and $a - b$ have the same sign (art. 35).

7. The three sides of a spherical triangle are together less than four right angles.

8. The three angles of a spherical triangle are greater than two right angles and less than six right angles (art. 31).

CHAPTER III.

ON FORMULÆ CONNECTING FUNCTIONS OF THE SIDES AND ANGLES OF A SPHERICAL TRIANGLE.

37. *The sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

Let ABC be a spherical triangle, and O the centre of the sphere.

Take any point D in OA, and from D let fall DE perpendicular to the plane BOC.

From E draw EF, EG perpendicular to OB, OC respectively, and join DF, DG, OE.

Because DE is perpendicular to the plane BOC, it is at right angles to every straight line meeting it in that plane (Euc. XI. *def.* 3).

Therefore the angles DEG, DEF, DEO are all right angles.

Therefore $DF^2 = DE^2 + EF^2 = OD^2 - OE^2 + OE^2 - OF^2 = OD^2 - OF^2$.

Thus the angle DFO is a right angle.

Similarly it may be shown that the angle DGO is a right angle.

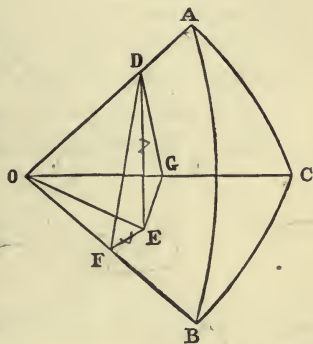
Now since DF, FE are each at right angles to OB, the angle DFE represents the inclinations of the planes AOB, BOC, that is, the angle B of the triangle ABC (art. 16).

Similarly it may be shown that DGE represents the angle C.

Thus $\quad DE = DF \sin. B = OD \sin. c \sin. B.$

Again, $\quad DE = DG \sin. C = OD \sin. b \sin. C.$

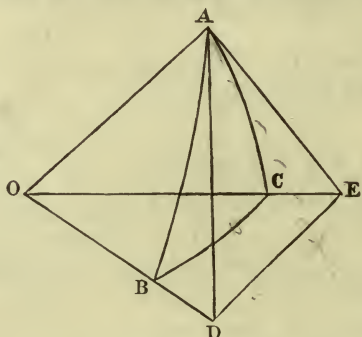
Therefore $\quad \sin. c \sin. B = \sin. b \sin. C;$

Or,
$$\frac{\sin. B}{\sin. b} = \frac{\sin. C}{\sin. c}.$$

Similarly it may be shown that

$$\frac{\sin. B}{\sin. b}=\frac{\sin. A}{\sin. a}.$$

Thus
$$\frac{\sin. A}{\sin. a}=\frac{\sin. B}{\sin. b}=\frac{\sin. C}{\sin. c}.$$

38. *To express the cosine of an angle of a spherical triangle in terms of sines and cosines of the sides.*

Let ABC be a spherical triangle, O the centre of the sphere. Let the tangent at A to the arc AB meet OB produced in D,

and let the tangent at A to the arc AC meet OC produced in E. Join DE.

Then the angle DAE measures the angle A of the triangle, and the angle DOE measures the side a.

And in the triangles ADE, ODE

$$DE^2=AD^2+AE^2-2AD\,.\,AE\cos.\,A.$$
$$DE^2=OD^2+OE^2-2OD\,.\,OE\cos.\,a.$$

Therefore, subtracting each side of the first equation from the second,

$$OD^2-AD^2+OE^2-AE^2+2AD\,.\,AE\cos.\,A$$
$$-2OD\,.\,OE\cos.\,a=0.$$

But since the angles OAD, OAE are right angles,

$$OA^2=OD^2-AD^2=OE^2-AE^2,$$

therefore $\quad 2OA^2+2AD\,.\,AE\cos.\,A-2OD\,.\,OE\cos.\,a=0.$

Therefore $\quad OD\,.\,OE\cos.\,a=OA^2+AD\,.\,AE\cos.\,A;$

and
$$\cos.\,a=\frac{OA}{OE}\cdot\frac{OA}{OD}+\frac{AE}{OE}\cdot\frac{AD}{OD}\cos.\,A.$$

Therefore $\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A$;

and $$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}.$$

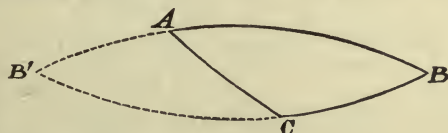
Similarly $$\cos. B = \frac{\cos. b - \cos. c \cos. a}{\sin. c \sin. a}$$

and $$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}.$$

Note.—In the construction of the figure given above, it has been assumed that the tangents AD, AE will meet the radii OB, OC respectively, when produced.

That this may be the case it is evident that each of the arcs AB, AC, which include the angle A, must be less than a quadrant.

It is easy, however, to show that if the theorem is true when AB, AC are each less than a quadrant, it is true also when either one, or both of these arcs, are greater than quadrants.

(1) Let one of the sides which contain the angle A, as AB, be greater than a quadrant.

Produce BA, BC to meet at B′.

Then BAB′, BCB′ are semicircles (art. 12).

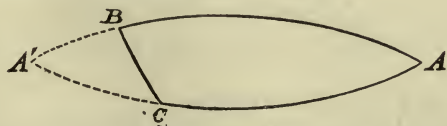
Let $AB' = c'$, $CB' = a'$.

In the triangle AB′C, since AB′, AC are each less than a quadrant, we have

$$\cos. a' = \cos. b \cos. c' + \sin. b \sin. c' \cos. B'AC.$$

And $a' = \pi - a$, $c' = \pi - c$, $B'AC = \pi - A$;

therefore $\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A$.

(2) Let both of the sides AB, AC, which contain the angle A, be greater than quadrants.

Produce AB, AC to meet at A′, and let $A'B = c'$, $A'C = b'$.

Then in the triangle A′BC we have

$$\cos. a = \cos. b' \cos. c' + \sin. b' \sin. c' \cos. A'.$$

And $b' = \pi - b,\ c' = \pi - c,\ A' = A;$

therefore $\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A.$

39. *To express the cosine of a side of a spherical triangle in terms of functions of the angles.*

Let ABC be any spherical triangle, and A′B′C′ its polar triangle, so that $A' = \pi - a$, $a' = \pi - A$, &c.

In the triangle A′B′C′ we have

$$\cos. A' = \frac{\cos. a' - \cos. b' \cos. c'}{\sin. b' \sin. c'} \text{ (art. 38).}$$

But $A' = \pi - a$, $a' = \pi - A$, $b' = \pi - B$, $c' = \pi - C$.

Substituting for A', a', &c. we have

$$\cos. (\pi - a) = \frac{\cos. (\pi - A) - \cos. (\pi - B) \cos. (\pi - C)}{\sin. (\pi - B) \sin. (\pi - C)};$$

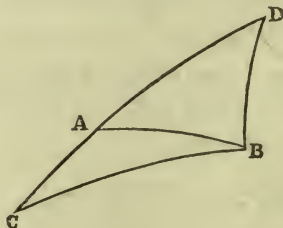
therefore
$$-\cos. a = \frac{-\cos. A - \cos. B \cos. C}{\sin. B \sin. C};$$

or,
$$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C}.$$

Similarly
$$\cos. b = \frac{\cos. B + \cos. C \cos. A}{\sin. C \sin. A};$$

$$\cos. c = \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B}.$$

40. *To show that in a spherical triangle* ABC,

$$\cot. a \sin. b = \cot. A \sin. C + \cos. b \cos. C.$$

Let ABC be a spherical triangle.

Produce the side CA to D, making AD equal to a quadrant, and join BD.

Then in the triangle BCD we have, by art. 38,

$$\cos. C = \frac{\cos. BD - \cos. (90^\circ + b) \cos. a}{\sin. (90^\circ + b) \sin. a}$$

$$= \frac{\cos. BD + \sin. b \cos. a}{\cos. b \sin. a};$$

therefore $\qquad \cos. BD = \cos. b \sin. a \cos. C - \sin. b \cos. a.$

Again, in the triangle ABD we have

$$\cos. (\pi - A) = \frac{\cos. BD - \cos. 90^\circ \cos. c}{\sin. 90^\circ \sin. c};$$

therefore $\qquad \cos. BD = -\cos. A \sin. c.$

Equating the two values of cos. BD we obtain

$$\cos. b \sin. a \cos. C - \sin. b \cos. a = -\cos. A \sin. c;$$

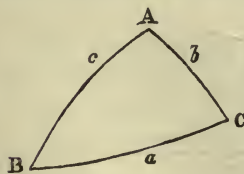
therefore $\cos. a \sin. b = \cos. A \sin. c + \cos. b \sin. a \cos. C.$

Dividing both sides of the equation by sin. a,

$$\cot. a \sin. b = \cos. A \frac{\sin. c}{\sin. a} + \cos. b \cos. C.$$

But $\frac{\sin. C}{\sin. A} = \frac{\sin. c}{\sin. a}$ (art. 37), and may therefore be substituted for it on the right hand side of the equation.

Therefore $\qquad \cot. a \sin. b = \cot. A \sin. C + \cos. b \cos. C.$

41. In the formula established in the preceding article it will be noticed that the four parts involved, namely, a, C, b, A, are *adjacent* parts of the triangle ABC, occurring in the order given as we go round the triangle. The relation obtained may be stated in words as follows, and in this form, perhaps, it is easier to retain the formula in the memory.

Cotangent of extreme side × sine of other side = cotangent of extreme angle × sine of other angle + product of cosines of the two middle parts.

Thus in the formula of art. 40, a is the extreme side, b falling between the angles A and C; A is the extreme angle, since

C is included between a and b. By taking in turn each side as the extreme side, we shall obtain six formulæ in all, each involving four parts of the triangle, adjacent to one another.

These formulæ are as follows:—

$$\text{Cot. } a \text{ sin. } b = \text{cot. A sin. C} + \text{cos. } b \text{ cos. C.}$$
$$\text{Cot. } a \text{ sin. } c = \text{cot. A sin. B} + \text{cos. } c \text{ cos. B.}$$
$$\text{Cot. } b \text{ sin. } a = \text{cos. B sin. C} + \text{cos. } a \text{ cos. C.}$$
$$\text{Cot. } b \text{ sin. } c = \text{cot. B sin. A} + \text{cos. } c \text{ cos. A.}$$
$$\text{Cot. } c \text{ sin. } a = \text{cot. C sin. B} + \text{cos. } a \text{ cos. B.}$$
$$\text{Cot. } c \text{ sin. } b = \text{cot. C sin. A} + \text{cos. } b \text{ cos. A.}$$

42. *To express the sine, cosine, and tangent of half an angle of a spherical triangle in terms of the sides.*

Since in the triangle ABC

$$\text{cos. A} = \frac{\text{cos. } a - \text{cos. } b \text{ cos. } c}{\text{sin. } b \text{ sin. } c} \quad \text{(art. 38)},$$

therefore
$$1 - \text{cos. A} = 1 - \frac{\text{cos. } a - \text{cos. } b \text{ cos. } c}{\text{sin. } b \text{ sin. } c}$$
$$= \frac{\text{sin. } b \text{ sin. } c - \text{cos. } a + \text{cos. } b \text{ cos. } c}{\text{sin. } b \text{ sin. } c}$$
$$= \frac{\text{cos. } (b-c) - \text{cos. } a}{\text{sin. } b \text{ sin. } c};$$

therefore
$$\text{sin.}^2 \frac{\text{A}}{2} = \frac{\text{sin. } \frac{1}{2}(a+b-c) \text{ sin. } \frac{1}{2}(a-b+c)}{\text{sin. } b \text{ sin. } c}.$$

Let $2s = a+b+c$, so that s is half the sum of the sides of the triangle.

Then
$$a+b-c = 2s-2c = 2(s-c),$$
$$a-b+c = 2s-2b = 2(s-b);$$

therefore
$$\text{sin.}^2 \frac{\text{A}}{2} = \frac{\text{sin. } (s-b) \text{ sin. } (s-c)}{\text{sin. } b \text{ sin. } c},$$

and
$$\text{sin. } \frac{\text{A}}{2} = \sqrt{\frac{\text{sin. } (s-b) \text{ sin. } (s-c)}{\text{sin. } b \text{ sin. } c}}$$

Again,

$$1 + \text{cos. A} = 1 + \frac{\text{cos. } a - \text{cos. } b \text{ cos. } c}{\text{sin. } b \text{ sin. } c} = \frac{\text{cos. } a - \text{cos. } (b+c)}{\text{sin. } b \text{ sin. } c};$$

therefore $\cos.^2 \frac{A}{2} = \frac{\sin. \frac{1}{2}(a+b+c) \sin. \frac{1}{2}(b+c-a)}{\sin. b \sin. c}$

$$= \frac{\sin. s \sin. (s-a)}{\sin. b \sin. c},$$

and $\cos. \frac{A}{2} = \sqrt{\frac{\sin. s \sin. (s-a)}{\sin. b \sin. c}}.$

From the expressions for $\sin. \frac{A}{2}$ and $\cos. \frac{A}{2}$ we obtain by division

$$\tan. \frac{A}{2} = \sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. s \sin. (s-a)}}.$$

The positive sign must in each case be given to the radicals in these equations, because, A being less than two right angles, $\frac{A}{2}$ must be less than 90°. Consequently the sine, cosine, and tangent of that angle will all be positive.

Proceeding in the same manner, it may be shown that

$$\sin. \frac{B}{2} = \sqrt{\frac{\sin. (s-a) \sin. (s-c)}{\sin. a \sin. c}},$$

$$\cos. \frac{B}{2} = \sqrt{\frac{\sin. s \sin. (s-b)}{\sin. a \sin. c}},$$

$$\sin. \frac{C}{2} = \sqrt{\frac{\sin. (s-a) \sin. (s-b)}{\sin. a \sin. b}},$$

$$\cos. \frac{C}{2} = \sqrt{\frac{\sin. s \sin. (s-c)}{\sin. a \sin. b}}.$$

43. *To express the haversine of an angle of a spherical triangle in terms of functions of the sides.*

Let ABC be a spherical triangle.

Then

$$\cos. A = \frac{\cos. a - \cos. b \cos c}{\sin. b \sin. c} \quad \text{art. 38)}.$$

Therefore $1 - \cos. A = 1 - \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c},$

$$\therefore \quad \text{vers. } A = \frac{\sin. b \sin. c - \cos. a + \cos. b \cos. c}{\sin. b \sin. c};$$

$$\therefore \quad 2 \text{ hav. } A = \frac{\cos.(b-c) - \cos. a}{\sin. b \sin. c}$$

$$= \frac{2 \sin. \frac{1}{2}(a+b-c) \sin. \frac{1}{2}(a-b+c)}{\sin. b \sin. c}.$$

But

$$\left.\begin{array}{l} \sin. \frac{1}{2}(a+b-c) = \sqrt{\text{hav. }(a+b-c)} \\ \sin. \frac{1}{2}(a-b+c) = \sqrt{\text{hav. }(a-b+c)} \end{array}\right\} \text{(Part I. art. 121)};$$

therefore

$$\text{hav. } A = \frac{\sqrt{\text{hav. }(a+b-c)}\sqrt{\text{hav. }(a-b+c)}}{\sin. b \sin. c}$$

$$= \sqrt{\text{hav. }(a+b-c)}\sqrt{\text{hav. }(a-b+c)} \text{ cosec. } b \text{ cosec. } c.$$

44. *To express the versine of any side of a spherical triangle in terms of functions of the other two sides, and of the angle included by those sides.*

Since in the triangle ABC we have

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c};$$

therefore

$$1 - \cos. A = 1 - \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c},$$

$$\therefore \quad \text{vers. } A = \frac{\sin. b \sin. c - \cos. a + \cos. b \cos. c}{\sin. b \sin. c};$$

$$\therefore \quad \cos.(b \sim c) - \cos. a = \sin. b \sin. c \text{ vers. } A,$$

and

$$-\cos. a = -\cos.(b \sim c) + \sin. b \sin. c \text{ vers. } A.$$

Adding unity to both sides we obtain

$$1 - \cos. a = 1 - \cos.(b \sim c) + \sin. b \sin. c \text{ vers. } A,$$

or,

$$\text{vers. } a = \text{vers. }(b \sim c) + \sin. b \sin. c \text{ vers. } A.$$

45. *To express the sine, cosine, and tangent of half a side of a spherical triangle in terms of functions of the angles.*

In the spherical triangle ABC we have

$$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C} \text{ (art. 39)};$$

therefore

$$1 - \cos. a = \frac{\sin. B \sin. C - \cos. A - \cos. B \cos. C}{\sin. B \sin. C},$$

and

$$2 \sin.^2 \frac{a}{2} = -\frac{\cos. A + \cos.(B+C)}{\sin. B \sin. C}$$

$$= -\frac{2 \cos. \frac{1}{2}(A+B+C) \cos. \frac{1}{2}(B+C-A)}{\sin. B \sin. C}.$$

Let $A+B+C=2S$, so that $B+C-A=2(S-A)$.

Then $$\sin.^2 \frac{a}{2} = -\frac{\cos. S \cos. (S-A)}{\sin. B \sin. C},$$

and $$\sin. \frac{a}{2} = \sqrt{-\frac{\cos. S \cos. (S-A)}{\sin. B \sin. C}}.$$

In a similar manner it may be shown that

$$\cos. \frac{a}{2} = \sqrt{\frac{\cos. (S-B) \cos. (S-C)}{\sin. B \sin. C}}.$$

$$\tan. \frac{a}{2} = \sqrt{-\frac{\cos. S \cos. (S-A)}{\cos. (S-B) \cos. (S-C)}}.$$

Similar expressions may be obtained for $\frac{b}{2}$ and $\frac{c}{2}$.

46. It should be observed that the values obtained in the preceding article are always *real*.

For by art. 31 the sum of the three angles of a spherical triangle lies always between π and 3π, so that cos. S is always a negative quantity.

Again, since two sides of a spherical triangle are always greater than the third, we have by the properties of the polar triangle

$$\pi - B + \pi - C > \pi - A.$$

Therefore $B + C - A < \pi$.

Hence $\frac{1}{2}(B + C - A)$ is less than $\frac{\pi}{2}$, and consequently cos. $(S-A)$ is a positive quantity, as also are cos. $(S-B)$, cos. $(S-C)$.

47. *To demonstrate Napier's Analogies.*

By art. 37 $\frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b}$. Let each of these ratios be equal to m.

Then we have, by algebra,

$$m = \frac{\sin. A + \sin. B}{\sin. a + \sin. b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$m = \frac{\sin. A - \sin. B}{\sin. a - \sin. b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now by art. 39

$$\cos. A + \cos. B \cos. C = \sin. B \sin. C \cos. a = m \sin. C \sin. b \cos. a.$$

and $$\cos. B + \cos. A \cos. C = \sin. A \sin. C \cos. b$$
$$= m \sin. C \sin. a \cos. b;$$

therefore, by addition,

$$(\cos. A + \cos. B)(1 + \cos. C) = m \sin. C \sin. (a+b) \quad . \quad . \quad (3)$$

Therefore by (1) and (3), when the two equations are divided one by the other, we obtain

$$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{\sin. a + \sin. b}{\sin. (a+b)} \quad \frac{1 + \cos. C}{\sin. C};$$

therefore $$\frac{2 \sin. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)}{2 \cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)}$$

$$= \frac{2 \sin. \frac{1}{2}(a+b) \cos. \frac{1}{2}(a-b)}{2 \sin. \frac{1}{2}(a+b) \cos. \frac{1}{2}(a+b)} \quad \frac{2 \cos.^2 \frac{C}{2}}{2 \sin. \frac{C}{2} \cos. \frac{C}{2}},$$

and $$\tan. \tfrac{1}{2}(A+B) = \frac{\cos. \frac{1}{2}(a-b)}{\cos. \frac{1}{2}(a+b)} \cot. \frac{C}{2} \quad . \quad . \quad . \quad (4)$$

Similarly from (2) and (3) we obtain

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \frac{\sin. a - \sin. b}{\sin. (a+b)} \quad \frac{1 + \cos. C}{\sin. C},$$

and $$\tan. \tfrac{1}{2}(A-B) = \frac{\sin. \frac{1}{2}(a-b)}{\sin. \frac{1}{2}(a+b)} \cot. \frac{C}{2} \quad . \quad . \quad . \quad (5)$$

By substituting $\pi - A$ for a, &c. in formulæ (4) (5), as explained in art. 28, these formulæ become

$$\tan. \tfrac{1}{2}(a+b) = \frac{\cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B)} \tan. \frac{c}{2} \quad . \quad . \quad . \quad . \quad (6)$$

$$\tan. \tfrac{1}{2}(a-b) = \frac{\sin. \frac{1}{2}(A-B)}{\sin. \frac{1}{2}(A+B)} \tan. \frac{c}{2} \quad . \quad . \quad . \quad . \quad (7)$$

These four equations (4) (5) (6) (7) are called from their discoverer *Napier's Analogies*. Equations (6) (7) may be established independently by commencing with the formulæ of art. 38, viz.

$$\cos. a - \cos. b \cos. c = \sin. b \sin. c \cos. A$$
$$\cos. b - \cos. a \cos. c = \sin. a \sin. c \cos. B.$$

To so deduce them for himself will be a useful exercise for the student.

48. In the relation (4) of the previous article cos. $\frac{1}{2}(a - b)$ and cot. $\frac{C}{2}$ must be positive quantities. It follows, therefore, that tan. $\frac{1}{2}(A + B)$, cos. $\frac{1}{2}(a + b)$ must necessarily have the same sign, that is, $\frac{1}{2}(A + B)$, $\frac{1}{2}(a + b)$ are either both less or both greater than a right angle. This is expressed by saying that $\frac{1}{2}(A + B)$ and $\frac{1}{2}(a + b)$ *are of the same affection.*

CHAPTER IV.

ON THE SOLUTION OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

49. The formulæ established in the preceding chapter will enable us in all cases when three of the six parts of a spherical triangle are given to determine the other parts. The three parts given may be either sides or angles. The several cases will be as follows:—

CASE I.

Three sides of a spherical triangle being given, to solve the triangle.

By art. 43 we have

$$\text{Hav. } A = \text{cosec. } b \text{ cosec. } c \sqrt{\text{hav.}(a + b - c) \text{ hav.}(a - b + c)};$$

therefore $\quad \text{L hav. } A = \text{L cosec. } b + \text{L cosec. } c$

$$+ \tfrac{1}{2} \text{L hav. } (a + \overline{b \sim c}) + \tfrac{1}{2} \text{L hav. } (a - \overline{b \sim c}) - 20.$$

In practice it is unnecessary to write down more than the mantissa of L cosec. b and L cosec. c. We need not then subtract 20 from the sum of the logarithms.

In the later editions of Inman's 'Nautical Tables' a table giving the values of half the logarithmic haversines is included, so that the trouble of dividing the tabular logarithms is avoided.

Having obtained one of the angles, as A, to determine B and C, we may if we please make use of the formulæ

$$\text{sin. } B = \frac{\text{sin. } b}{\text{sin. } a} \text{ sin. } A; \text{ sin. } C = \frac{\text{sin. } c}{\text{sin. } a} \text{ sin. } A.$$

On account, however, of the ambiguities which attend the use of these formulæ (which will be considered later in arts. 52,

53), it is better to determine B and C by the haversine formula, as in the case of A.

CASE II.

50. *Having given two sides of a spherical triangle, and the angle included by those sides, to determine the other parts.*

Let b, c, A be the parts given.

We shall first determine the side a.

By art. 44

$$\text{vers. } a = \text{vers. } (b \sim c) + \text{sin. } b \text{ sin. } c \text{ vers. A}.$$

Since sin. b sin. c vers. A is never greater than vers. A, for sin. b, sin. c cannot either of them be greater than unity, an angle θ may always be found such that

$$\text{vers. } \theta = \text{sin. } b \text{ sin. } c \text{ vers. A}.$$

So that vers. a = vers. $(b \sim c)$ + vers. θ.

To find θ.

Since $\quad$ vers. θ = sin. b sin. c vers. A,

therefore $\quad$ hav. θ = sin. b sin. c hav. A.

By means of the table of logarithmic haversines we may then find θ.

Thus $\quad$ L hav. θ = L sin. b + L sin. c + L hav. A − 20.

Then since $\quad$ vers. a = vers. $(b \sim c)$ + vers. θ,

therefore $$\frac{\text{tab. vers. } a}{1{,}000{,}000} = \frac{\text{tab. vers. } (b \sim c)}{1{,}000{,}000} + \frac{\text{tab. vers. } \theta}{1{,}000{,}000};$$

and $\quad$ tab. vers. a = tab. vers. $(b \sim c)$ + tab. vers. θ.

Having now the three sides of the triangle, we may proceed to determine the remaining angles by the haversine method, as already explained.

51. When two sides and the included angle are given, we may if we please determine the remaining angles directly from the data. Thus, if b, c, A are given, we have by Napier's Analogies (art. 47)

$$\text{tan. } \tfrac{1}{2}(\text{B} + \text{C}) = \frac{\text{cos. } \frac{1}{2}(b - c)}{\text{cos. } \frac{1}{2}(b + c)} \text{ cot. } \frac{\text{A}}{2};$$

$$\text{tan. } \tfrac{1}{2}(\text{B} - \text{C}) = \frac{\text{sin. } \frac{1}{2}(b - c)}{\text{sin. } \frac{1}{2}(b + c)} \text{ cot. } \frac{\text{A}}{2}.$$

From these formulæ we obtain the values of $\frac{1}{2}$ (B + C) and $\frac{1}{2}$ (B − C), whence by addition and subtraction the values of B, C will be found.

In practical calculations it is probably better to proceed as in art. 50, and first determine the third side, although in solving examples it will be found a useful exercise to solve the same triangle by each method.

Case III.

52. *Having given in a spherical triangle two sides and the angle opposite to one of them, to determine the other parts.*

Let a, b, A be the parts given.

Since $\sin. B = \dfrac{\sin. b}{\sin. a} \sin. A$,

therefore $L \sin. B = L \sin. b + L \sin. A - L \sin. a$.

Since the angle B is determined from its sine there will be two angles less than 180°, corresponding to the tabular logarithm L sin. B, and it will sometimes happen that there are two triangles having the given parts a, b, A.

It becomes necessary, therefore, whenever the above formula is used in obtaining an angle of a spherical triangle, to ascertain whether the value of B which is required is greater or less than 90°, or whether both values are admissible, so that two triangles are possible which satisfy the data of the problem.

In the decision of this point we must be guided by the consideration that in any spherical triangle *the greater angle must be subtended by the greater side* (arts. 34, 35). From this property it follows that if a be greater than b, A must be greater than B; so that $a - b$, A − B must always have the same sign.

For suppose that we have given $a = 80°$, $b = 70°$, $A = 40°$; we find from the tables that L sin. B = 9·787702, so that B = 37° 50′ or 142° 10′. Since b is less than a, B must be less than A, and the second of these values must therefore be rejected. Again, let $a = 70°$, $b = 100°$, $A = 50°$. Here L sin. B = 9·904619, and B = 53° 24′ or 126° 36′. Since b is greater than a, B must be greater than A; a condition which is satisfied

whichever value of B be taken. The case is therefore an ambiguous one, and two triangles may be constructed having the given parts.

As will be shown later, the cases in which only *one* solution is to be expected are those in which a, the side opposite to the angle given, lies between b and $\pi - b$.

Having now two sides, and the angles opposite to these sides, we may proceed to determine the remaining parts c, C by Napier's Analogies.

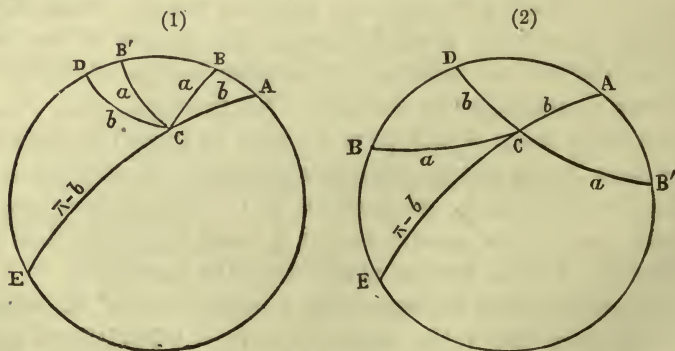
Thus, to find c we have, by art. 47,

$$\tan.\frac{c}{2} = \frac{\cos.\frac{1}{2}(A+B)}{\cos.\frac{1}{2}(A-B)}\tan.\tfrac{1}{2}(a+b);$$

while C is given by the formula

$$\tan.\frac{C}{2} = \frac{\cos.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}(a+b)}\cot.\tfrac{1}{2}(A+B).$$

53. *To show that in a spherical triangle, in which* a, b, A *are given, there will be only one solution if* a *lies between* b *and* $\pi - b$.

Let ADE be a great circle, and at the point A let a second great circle ACE cut ADE at the given angle A, supposed less than a right angle, and let the two circles intersect again in E.

Then AE is a semicircle (art. 12).

From AE cut off AC equal to the given side b, supposed less than $\frac{\pi}{2}$.

Therefore CE is equal to $\pi - b$.

With C as centre, and radius equal to b, describe the arc of a small circle, cutting ADE in D, so that $CD = CA = b$.

First let a, the side opposite to the given angle, be less than b, and consequently less than $\pi - b$ also.

With centre C, and radius equal to the given value a, describe the arc of a small circle.

This will cut the circle ADE in two points, as B, B′, between A and D, as shown in figure (1).

Two triangles will then be formed, viz. ABC, AB′C, each of which has the given parts a, b, A, and there is therefore a double solution.

Next, let a lie between b and $\pi - b$ in value.

The same construction being made, the small circle described with radius equal to a will cut the circle ADE in two points B, B′ as before, but these points will lie on opposite sides of the point E, as shown in figure (2).

There will therefore be only one triangle, viz. ABC, having the given parts, since in the triangle AB′C, the value of the angle at A will be $\pi - A$ instead of A.

Note.—In the above investigation we have taken as granted that the value of any arc of a great circle drawn through C, and cutting the circle ADE between D and E, is intermediate in value between CD and CE. The point may be for the present assumed. At a later stage, when the student has made himself acquainted with the relations between the sides of right angled spherical triangles, by dropping a perpendicular from C upon the circle ADE he may easily establish it for himself.

Moreover, although we have selected for illustration the particular case in which the side b and the angle A are each less than $\frac{\pi}{2}$, the statement of limitations is true generally, and may be shown to hold for all values of b and A.

Case IV.

54. *Having given two angles, and the side opposite to one of them, to determine the other parts.*

Let a, A, B be the parts given.

Then $\sin. b = \frac{\sin. B}{\sin. A} \sin. a$.

The same ambiguities will arise in this case as in the preceding one, and we must be guided by the same considerations.

And it is to be remembered that when a, A, B are given, *one* solution only is possible whenever A lies between B and $\pi - B$.

To find c and C we make use of Napier's Analogies, as in art. 52.

Case V.

55. *Having given two angles, and the side included between them, to find the other parts.*

Let A, B, c be the parts given.

We may, if we please, make use of Napier's Analogies to find a, b.

Thus,
$$\tan. \tfrac{1}{2}(a+b) = \frac{\cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B)} \tan. \frac{c}{2};$$
$$\tan. \tfrac{1}{2}(a-b) = \frac{\sin. \frac{1}{2}(A-B)}{\sin. \frac{1}{2}(A+B)} \tan. \frac{c}{2}.$$

It is, however, not unusual in this case to resort to the polar triangle.

Thus, since in the primitive triangle A, B, c are known, we have in the polar triangle a', b', C$'$, two sides and the included angle.

From these data we may obtain c', as shown in art. 50; then, having three sides, we may determine A$'$, B$'$ by the haversine formula used in art. 49.

Thus the six parts of the polar triangle are completely determined, and the supplements of these parts, which are the elements of the primitive triangle, are therefore known also.

Case VI.

56. *Having given the three angles of a spherical triangle, to find the three sides.*

In art. 45 are established certain expressions for $\sin. \frac{a}{2}$, &c., in terms of functions of the angles, which may, if we please

be made available. Here again, however, it is preferable to resort to the polar triangle. Thus A, B, C being known in the triangle ABC, a', b', c', the three sides of the polar triangle are known also.

Hence by art. 49 we may determine the three angles A', B', C', and consequently the supplements of these angles, the three sides a, b, c.

CHAPTER V.

ON THE SOLUTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

57. If one angle of a spherical triangle be a right angle, the triangle may be solved by processes simpler than those given in the preceding chapter.

In Chapter III. we have established a series of fundamental formulæ connecting functions of the sides and angles of a spherical triangle, each formula involving *four* parts of the triangle.

Let one of the angles, as C, be a right angle.

Then, if we select all those equations which contain C, and substitute for the sine and other functions of C their actual values, we shall obtain a series of equations connecting *three* parts of the given triangle.

And from these equations, as will be shown, when two parts of a triangle are given, any other may be determined by the addition or subtraction of two logarithms.

If, then, C be taken as the right angle, the formulæ which we shall require will be as follows:

$$\left.\begin{array}{l}\text{Sin. } a \text{ sin. C} = \text{sin. } c \text{ sin. A} \\ \text{sin. } b \text{ sin. C} = \text{sin. } c \text{ sin. B}\end{array}\right\} \quad (a) \text{ art. 37.}$$

$$\text{Cos. } c = \text{cos. } a \text{ cos. } b + \text{sin. } a \text{ sin. } b \text{ cos. C} \quad (b) \text{ art. 38.}$$

$$\left.\begin{array}{l}\text{Cot. } a \text{ sin. } b = \text{cot. A sin. C} + \text{cos. } b \text{ cos. C} \\ \text{cot. } b \text{ sin. } a = \text{cot. B sin. C} + \text{cos. } a \text{ cos. C} \\ \text{cot. } c \text{ sin. } a = \text{cot. C sin. B} + \text{cos. } a \text{ cos. B} \\ \text{cot. } c \text{ sin. } b = \text{cot. C sin. A} + \text{cos. } b \text{ cos. A}\end{array}\right\} \quad (c) \text{ art. 40.}$$

$$\left.\begin{aligned}\text{Cos. } c \text{ sin. A sin. B} &= \text{cos. C} + \text{cos. A cos. B}\\ \text{cos. } a \text{ sin. B sin. C} &= \text{cos. A} + \text{cos. B cos. C}\\ \text{cos. } b \text{ sin. A sin. C} &= \text{cos. B} + \text{cos. A cos. C}\end{aligned}\right\} (d) \text{ art. 39.}$$

If C = 90°, we shall derive the following relations from those given:

From (*a*)
$$\left.\begin{aligned}\text{sin. } a &= \text{sin. A sin. } c\\ \text{sin. } b &= \text{sin. B sin. } c\end{aligned}\right\}$$

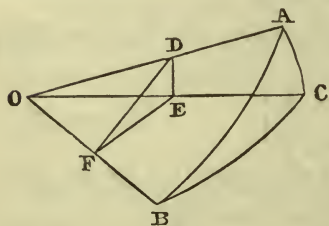
(*b*)
$$\text{cos. } c = \text{cos. } a \text{ cos. } b$$

(*c*)
$$\left.\begin{aligned}\text{sin. } b &= \text{cot. A tan. } a\\ \text{sin. } a &= \text{cot. B tan. } b\\ \text{cos. B} &= \text{tan. } a \text{ cot. } c\\ \text{cos. A} &= \text{tan. } b \text{ cot. } c\end{aligned}\right\}$$

(*d*)
$$\left.\begin{aligned}\text{cos. } c &= \text{cot. A cot. B}\\ \text{cos. A} &= \text{cos. } a \text{ sin. B}\\ \text{cos. B} &= \text{cos. } b \text{ sin. A}\end{aligned}\right\}$$

Thus we have in all ten formulæ, each involving three of the five parts a, b, c, A, B. And since ten is the total number of combinations which can be formed by five quantities taken three together, it follows that when two of these quantities are given we may always determine the third by one or other of the above equations.

58. The formulæ for right angled triangles may be established independently as follows:

Let ABC be a spherical triangle, having a right angle at C, and let O be the centre of the sphere. From any point D in OA draw DE perpendicular to OC, and from E draw EF perpendicular to OB, and join DF.

Then DE is perpendicular to EF, because the plane AOC is perpendicular to the plane BOC (Euc. XI. *def.* 4).

And

$$DF^2 = DE^2 + EF^2 = OD^2 - OE^2 + OE^2 - OF^2 = OD^2 - OF^2.$$

Therefore the angle OFD is a right angle, so that the angle

EFD represents the inclination of the planes AOB, BOC, that is, it represents the angle B (art. 16).

Then $\frac{DE}{OD} = \frac{DE}{DF}\frac{DF}{OD}$, or $\sin. b = \sin. B \sin. c$.
By a similar construction it may be shown that $\sin. a = \sin. A \sin. c$. (1)

Again, $\frac{OF}{OD} = \frac{OF}{OE}\frac{OE}{OD}$, or $\cos. c = \cos. a \cos. b$. (2)

And $\frac{EF}{OE} = \frac{EF}{DE}\frac{DE}{OE}$, or $\sin. a = \cot. B \tan. b$.
Similarly it may be shown that $\sin. b = \cot. A \tan. a$. (3)

And $\frac{EF}{DF} = \frac{EF}{OF}\frac{OF}{DF}$, or $\cos. B = \tan. a \cot. c$.
Similarly it may be shown that $\cos. A = \tan. b \cot. c$. (4)

Multiplying together the two formulæ in (3) we obtain $\cot. A \cot. B \tan. a \tan b = \sin. a \sin. b$;

therefore $\cot. A \cot. B = \cos. a \cos. b = \cos c$. (5)

Again, by (4)

$$\cos. B = \tan. a \cot. c = \frac{\sin. a \cos. c}{\cos. a \sin. c}.$$

But by the second equation of (1) $\sin. A = \frac{\sin. a}{\sin. c}$.

Therefore $\cos. B = \frac{\sin. A \cos. c}{\cos. a}$;

and by (2) $\cos. b = \frac{\cos. c}{\cos. a}$;

therefore $\cos. B = \sin. A \cos. b$.
In the same way we may show that
$\cos. A = \sin. B \cos. a$. (6)

59. From the two formulæ (3) of art. 58

$$\sin. a = \cot. B \tan. b;$$
$$\sin. b = \cot. A \tan. a.$$

Since a, b are each less than 180°, it follows that the expressions $\cot. B \tan. b$ and $\cot. A \tan. a$ must both be positive.

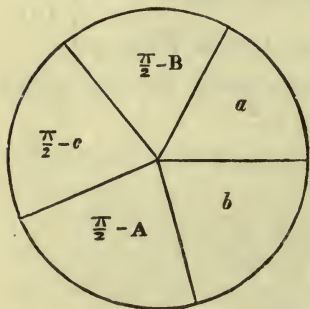
And since A, a are each less than 180°, A, a must be either both less or both greater than 90°; that is, A, a are of the same affection.

Similarly, B, b are of the same affection.

60. The ten formulæ which we have given for right angled triangles are comprehended under two rules, called, after their inventor, *Napier's Rules of Circular Parts.*

These rules may be explained as follows :

Let ABC be a spherical triangle in which C is the right angle.

Then, excluding C, the circular parts are the two sides including the right angle, and the complements of the hypothenuse and of the angles A and B.

These five parts may be ranged round a circle in the order in which they occur with respect to the triangle.

Any one of these parts may be selected, and may be called the *middle part*; then the two parts next to it are called the *adjacent* parts, and the other two parts are called the *opposite* parts.

Napier's rules are two in number.

Sine of the middle part = product of tangents of adjacent parts.

Sine of the middle part = product of cosines of opposite parts.

The occurrence of the vowel i in the words '*sine*,' '*middle*,' of the vowel a in '*tangent*' and '*adjacent*,' and of the vowel o in the words '*cosine*' and '*opposite*' renders these rules easy to remember.

61. By taking in detail each of the five parts as the middle part, and writing down the equations furnished by these two rules, we shall obtain the same ten equations established in art. 57.

Thus, commencing with c,

$$\text{Sin.}\left(\frac{\pi}{2}-c\right) = \tan.\left(\frac{\pi}{2}-\text{A}\right)\tan.\left(\frac{\pi}{2}-\text{B}\right)$$

$$\therefore \cos.\, c = \cot.\text{A} \cot.\,\text{B}.$$

$$\text{Sin.}\left(\frac{\pi}{2}-c\right) = \cos.\, a \cos.\, b \qquad \therefore \cos.\, c = \cos.\, a \cos.\, b.$$

$$\text{Sin.}\left(\frac{\pi}{2}-\text{B}\right) = \tan.a \tan.\left(\frac{\pi}{2}-c\right) \qquad \therefore \cos.\,\text{B} = \tan.\, a \cot.\, c.$$

$$\text{Sin.}\left(\frac{\pi}{2}-\text{B}\right) = \cos.b \cos.\left(\frac{\pi}{2}-\text{A}\right) \qquad \therefore \cos.\,\text{B} = \cos.b \sin.\,\text{A}.$$

$$\text{Sin.}a = \tan.\left(\frac{\pi}{2}-\text{B}\right)\tan.b \qquad \therefore \sin.\, a = \cot.\,\text{B} \tan.\, b.$$

$$\text{Sin.}a = \cos.\left(\frac{\pi}{2}-\text{A}\right)\cos.\left(\frac{\pi}{2}-c\right) \qquad \therefore \sin.\, a = \sin.\,\text{A} \sin.\, c.$$

$$\text{Sin.}b = \tan.\left(\frac{\pi}{2}-\text{A}\right)\tan.a \qquad \therefore \sin.\, b = \cot.\,\text{A} \tan.a.$$

$$\text{Sin.}b = \cos.\left(\frac{\pi}{2}-\text{B}\right)\cos.\left(\frac{\pi}{2}-c\right) \qquad \therefore \sin.\, b = \sin.\,\text{B} \sin.\, c.$$

$$\text{Sin.}\left(\frac{\pi}{2}-\text{A}\right) = \tan.b \tan.\left(\frac{\pi}{2}-c\right) \qquad \therefore \cos.\,\text{A} = \tan.\, b \cot.\, c.$$

$$\text{Sin.}\left(\frac{\pi}{2}-\text{A}\right) = \cos.a \cos.\left(\frac{\pi}{2}-\text{B}\right) \qquad \therefore \cos.\,\text{A} = \cos.a \sin.\,\text{B}.$$

62. The method of applying Napier's Rules is shown in the following example.

Let ABC be a spherical triangle, having the angle C a right angle, and let the parts given be b, A. It is required to find the other parts. Let A be greater than 90°, and b less than 90°.

First to find a.

Referring to the circle of art. 60, we see that of the three parts a, b, $\frac{\pi}{2}-$ A, b is the middle part, and that it has a, $\frac{\pi}{2}-$ A for its adjacent parts.

$$\text{Therefore } \sin.b = \tan.\,a \tan.\left(\frac{\pi}{2}-\text{A}\right), \text{ or } \overset{-}{\tan.\, a} = \overset{+}{\sin.\, b}\, \overset{-}{\tan.\,\text{A}}.$$

Thus $\quad$ L tan. a = L sin. b + L tan. A − 10.

To determine whether a is greater or less than 90°, we must be guided by the algebraical sign of tan. a, which depends, of course, upon the sign due to the product which forms the right-hand side of the equation. We must therefore be careful to write over each function the appropriate sign, as shown above.

To find c, we may if we please make use of the formulæ connecting a, b, c, now that the value of a is determined.

It is better, however, to solve the triangle completely by means of the two parts first given, since by making use of the side a, any error which has been made in calculating a will vitiate the results for c and B also. Moreover, the labour of calculating is slightly simplified when the same two parts are made use of.

We have then to consider next the three parts b, $\frac{\pi}{2}-c$, $\frac{\pi}{2}-\text{A}$.

Here $\frac{\pi}{2}-\text{A}$ is the middle part, b, $\frac{\pi}{2}-c$ are the adjacent parts.

Hence $$\sin.\left(\frac{\pi}{2}-\text{A}\right) = \tan. b \tan.\left(\frac{\pi}{2}-c\right), \text{ or}$$

$$\overset{-}{\cot.\ c} = \overset{-}{\cos.\ \text{A}}\ \overset{+}{\cot.\ b}.$$

Therefore L cot. c = L cos. A + L cot. b − 10, and cot. c being negative, c is greater than 90°, so that the value required is the supplement of the angle found in the tables.

To find B we have $\frac{\pi}{2}-\text{B}$ as the middle part, $\frac{\pi}{2}-\text{A}$ and b as the opposite parts.

Then $$\sin.\left(\frac{\pi}{2}-\text{B}\right) = \cos. b \cos.\left(\frac{\pi}{2}-\text{A}\right),$$

whence $$\overset{+}{\cos.\ \text{B}} = \overset{+}{\cos.\ b}\ \overset{+}{\sin.\ \text{A}},$$

and $$\text{L} \cos. \text{B} = \text{L} \cos. b + \text{L} \sin. \text{A} - 10.$$

63. When, as in the example of the preceding article, the unknown parts of the triangle are determined by means of the tangent, cotangent, or cosine, there can be no ambiguity, as the algebraic sign of the function from which the part required is determined indicates whether the latter is greater or less than 90°.

But if the required part is determined from its sine, there will be two values, each less than 180°, which will satisfy the equation, so that sometimes two triangles may be found possessing the given data.

The five equations in which the unknown part is found from the sine may be divided into two classes, as follows:—

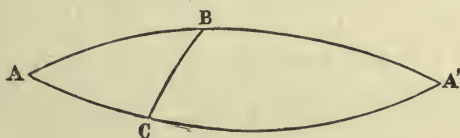
(1). $\sin. c = \dfrac{\sin. a}{\sin. A}$, $\sin. b = \dfrac{\tan. a}{\tan. A}$, $\sin. B = \dfrac{\cos. A}{\cos. a}$,

in each of which the two given parts are a side and its opposite angle.

(2). $\sin. a = \sin. c \sin. A$, $\sin. A = \dfrac{\sin. a}{\sin. c}$.

In the first class the solution is a double one, as may be shown.

Thus, let ABC be a spherical triangle in which C is a right angle.

Then if the sides AB, AC are produced to A′, a second

triangle A′BC is obtained, having the side BC in common with the triangle ABC, and the angle A′ equal to the angle A; but the side A′B=180°−AB, A′C=180°−AC, and the angle A′BC =180°−ABC.

The equations in the first class afford a double solution, since in each case the parts given are a, A, which are common to both triangles ABC, A′BC.

In the second class, however, there is but one solution, and, as shown in art. 59, a will be greater or less than 90°, according as A is greater or less than 90°, and conversely.

CHAPTER VI.

ON THE SOLUTION OF QUADRANTAL SPHERICAL TRIANGLES.

64. In the spherical triangle ABC let one side be a quadrant. In this case also the fifteen fundamental formulæ, established in Chapter III., may be simplified so as to furnish at once logarithmic expressions for the solution of the triangle similar to those already obtained for right-angled triangles.

Thus, let us suppose that the side $c = 90°$.

Then, as before, there will be ten formulæ containing the side c, as follows :

$$\left.\begin{array}{l}\text{sin. A sin. } c = \text{sin. C sin. } a \\ \text{sin. B sin. } c = \text{sin. C sin. } b\end{array}\right\} \quad (a) \text{ art. 37.}$$

$$\left.\begin{array}{l}\text{cos. } a = \text{cos. } b \text{ cos. } c + \text{sin. } b \text{ sin. } c \text{ cos. A} \\ \text{cos. } b = \text{cos. } a \text{ cos. } c + \text{sin. } a \text{ sin. } c \text{ cos. B} \\ \text{cos. } c = \text{cos. } a \text{ cos. } b + \text{sin. } a \text{ sin. } b \text{ cos. C}\end{array}\right\} \quad (b) \text{ art. 38.}$$

$$\left.\begin{array}{l}\text{cot. } a \text{ sin. } c = \text{cot. A sin. B} + \text{cos. } c \text{ cos. B} \\ \text{cot. } c \text{ sin. } a = \text{cot. C sin. B} + \text{cos. } a \text{ cos. B} \\ \text{cot. } b \text{ sin. } c = \text{cot. B sin. A} + \text{cos. } c \text{ cos. A} \\ \text{cot. } c \text{ sin. } b = \text{cot. C sin. A} + \text{cos. } b \text{ cos. A}\end{array}\right\} \quad (c) \text{ art. 40.}$$

$$\text{cos. C} = \text{sin. A sin. B cos. } c - \text{cos. A cos. B} \qquad (d) \text{ art. 39.}$$

In these formulæ, if we substitute for sin. c and cos. c the values of sin. 90° and cos. 90°, we shall obtain ten formulæ as follows :—

$$\left.\begin{array}{l}\text{sin. A} = \text{sin. } a \text{ sin. C} \\ \text{sin. B} = \text{sin. } b \text{ sin. C}\end{array}\right\} \quad (a)$$

$$\left.\begin{array}{l}\text{cos. } a = \text{sin. } b \text{ cos. A} \\ \text{cos. } b = \text{sin. } a \text{ cos. B} \\ \text{cos. C} = -\text{ cot. } a \text{ cot. } b\end{array}\right\} \quad (b)$$

$$\left.\begin{array}{l}\text{cot. } a = \text{cot. A sin B} \\ \text{cos. } a = -\text{ tan. B cot C} \\ \text{cot. } b = \text{cot. B sin. A} \\ \text{cos. } b = -\text{ tan. A cot. C}\end{array}\right\} \quad (c)$$

$$\text{cos. C} = -\text{ cos. A cos. B} \qquad (d)$$

65. The ten formulæ for quadrantal triangles may, if we please, be obtained from those established in art. 57 for right-angled triangles, by making use of the polar triangle.

Thus, let ABC be a spherical triangle having the angle C a right angle. Then if A′B′C′ be the polar triangle to ABC, it has the side c' a quadrant.

In the triangle ABC, if we take the first three formulæ established in art. 57, we have

$$\sin. a = \sin. A \sin. c,$$
$$\sin. b = \sin. B \sin. c,$$
$$\cos. c = \cos. a \cos. b.$$

And since $a = \pi - A'$, $b = \pi - B'$, &c., the formulæ may be written

$$\sin. (\pi - A') = \sin. (\pi - a') \sin. (\pi - C');$$
$$\sin. (\pi - B') = \sin. (\pi - b') \sin. (\pi - C');$$
$$\cos. (\pi - C') = \cos. (\pi - A') \cos. (\pi - B').$$

Therefore
$$\sin. A' = \sin. a' \sin. C';$$
$$\sin. B' = \sin. b' \sin. C';$$
$$\cos. C' = -\cos. A' \cos. B';$$

results which agree with those obtained by other methods in art. 64.

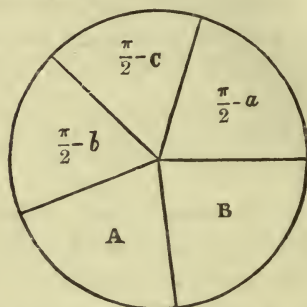
Proceeding in the same manner, we might deduce the remainder of the formulæ for quadrantal triangles from those established for right-angled triangles in art. 57.

66. The rules of circular parts explained in the previous chapter for right-angled triangles may be utilised for quadrantal triangles under certain modifications.

The rules for quadrantal triangles differ from those for right-angled triangles in two particulars:—

(1). *The elements of the triangle of which we have to take the complements are the sides* a, b, *and the angle* C, c *being the quadrant.*

(2). *Whenever the two adjacent, or the two opposite parts, are both sides, or both angles, the sign* — *must be attached to the product.*

The circular parts for a triangle ABC when c is a quadrant, will therefore be arranged as follows:

And by taking each in turn as the middle part, and applying the two principles—

sine of the middle part = the product of the tangents of the adjacent parts;

sine of the middle part = the product of the cosines of the opposite parts,

the ten formulæ of art. 64 will be obtained.

67. In the quadrantal, as in the right-angled triangle, an ambiguity arises when the two parts given are a side and the opposite angle, so that two solutions are then possible.

In all other cases in which an apparent ambiguity presents itself, the proper value of the required part may be determined from the consideration that in a quadrantal triangle an angle and the side opposite to it must be either both greater or both less than 90°.

That in a triangle ABC which has the side c a quadrant, A, a, and B, b must be respectively of the same affection will easily appear from a consideration of the two formulæ (c) of art. 64, viz.

$$\left.\begin{array}{l}\text{cot. } a = \text{cot. A sin. B} \\ \text{cot. } b = \text{cot. B sin. A}\end{array}\right\}$$

68. Appended is a collection of exercises, of varying degrees of difficulty, upon the subject matter of the foregoing chapters. They have for the most part been taken from the papers of questions set in the examinations for rank of lieutenant during the last five years.

FORMULÆ OF REFERENCE (III.).

In any spherical triangle

(1) $\frac{\sin. A}{\sin. a}=\frac{\sin. B}{\sin. b}=\frac{\sin. C}{\sin. c}$ (art. 37)

(2) $\cos. A=\frac{\cos. a-\cos. b \cos. c}{\sin. b \sin. c}$ (art. 38)

(3) $\cos. a=\frac{\cos. A+\cos. B \cos. C}{\sin. B \sin. C}$ (art. 39)

(4) $\cot. a \sin. b=\cot. A \sin. C+\cos. b \cos. C$ (art. 40)

(5) $\sin. \frac{A}{2}=\sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. b \sin. c}}$, where $2s=a+b+c$

(6) $\cos. \frac{A}{2}=\sqrt{\frac{\sin. s \sin. (s-a)}{\sin. b \sin. c}}$,, ,,

(7) $\tan. \frac{A}{2}=\sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. s \sin. (s-a)}}$,, ,,

(art. 42)

(8) $\text{hav. } A=\sqrt{\text{hav. } (a+b-c)}\sqrt{\text{hav. } (a-b+c)} \text{ cosec. } b \text{ cosec } c$ (art. 43)

(9) $\text{vers. } a=\text{vers. } (b\sim c)+\sin. b \sin. c \text{ vers. } A$ (art. 44)

(10) $\sin. \frac{a}{2}=\sqrt{-\frac{\cos. S \cos. (S-A)}{\sin. B \sin. C}}$, where $2S=A+B+C$

(11) $\cos. \frac{a}{2}=\sqrt{\frac{\cos. (S-B) \cos. (S-C)}{\sin. B \sin. C}}$,, ,,

(12) $\tan. \frac{a}{2}=\sqrt{-\frac{\cos. S \cos. (S-A)}{\cos. (S-B) \cos. (S-C)}}$,, ,,

(art. 45)

(13) $\tan. \frac{1}{2} (A+B)=\frac{\cos. \frac{1}{2} (a-b)}{\cos. \frac{1}{2} (a+b)} \cot. \frac{C}{2}$

$\tan. \frac{1}{2} (A-B)=\frac{\sin. \frac{1}{2} (a-b)}{\sin. \frac{1}{2} (a+b)} \cot. \frac{C}{2}$ . . . (art. 47)

(14) $\tan. \frac{1}{2} (a+b)=\frac{\cos. \frac{1}{2} (A-B)}{\cos. \frac{1}{2} (A+B)} \tan. \frac{c}{2}$

$\tan. \frac{1}{2} (a-b)=\frac{\sin. \frac{1}{2} (A-B)}{\sin. \frac{1}{2} (A+B)} \tan. \frac{c}{2}$ . . . (art. 47)

In any spherical triangle having C a right angle

(15)
$$\left.\begin{aligned}
&\sin. a = \sin. A \sin. c\\
&\sin. b = \sin. B \sin. c\\
&\cos. c = \cos. a \cos. b\\
&\sin. b = \cot. A \tan. a\\
&\sin. a = \cot. B \tan. b\\
&\cos. B = \tan. a \cot. c\\
&\cos. A = \tan. b \cot. c\\
&\cos. c = \cot. A. \cot. B\\
&\cos. A = \cos. a \sin. B\\
&\cos. B = \cos. b \sin. A
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(art. 57)}$$

In any spherical triangle having c a quadrant

(16)
$$\left.\begin{aligned}
&\sin. A = \sin. a \sin. C\\
&\sin. B = \sin. b \sin. C\\
&\cos. a = \sin. b \cos. A\\
&\cos. b = \sin. a \cos. B\\
&\cos. C = -\cot. a \cot. b\\
&\cot. a = \cot. A \sin. B\\
&\cos. a = -\tan. B \cot. C\\
&\cot. b = \cot. B \sin. A\\
&\cos. b = -\tan. A \cot. C\\
&\cos. C = -\cos. A \cos. B
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(art. 64)}$$

MISCELLANEOUS EXAMPLES.

1. Show that in any equilateral spherical triangle ABC the following relations hold:

(*a*) $\text{Cos. A}=\dfrac{\cos. a}{1+\cos. a}$.

(*b*) $2\cos.\dfrac{a}{2}\ \sin.\dfrac{A}{2}=1$.

(*c*) $\tan.^2\dfrac{a}{2}=1-2\cos. A$.

(*d*) $\text{Sec. A}-\sec. a=1$.

(*e*) $1+2\cos. a=\cot.^2\dfrac{A}{2}$.

2. Show that in a spherical triangle which has each of its sides equal to 60° the cosines of each of the angles are equal to $\frac{1}{3}$.

3. If D is the middle point of BC, and AD be joined, prove that

$$\cos. b+\cos. c=2\cos. \text{AD}\cos. \frac{a}{2}.$$

4. If $b+c=90°$, prove that

(*a*) $\text{Cos. } a=\sin. 2c\cos.^2\dfrac{A}{2}$.

(*b*) $(\text{Cos. } c+\sin. c)\sin. A=2\cos.^2\dfrac{a}{2}\sin. (B+C)$.

5. Show that the perpendicular from any vertex of a spherical triangle upon the opposite side divides the angle and that side into two parts whose tangents have the same ratio.

6. If A be one of the base angles of an isosceles spherical triangle, whose vertical angle is $90°$, and a the side opposite, show that cos. $a=$cot. A, and determine the limits between which A must lie.

7. If p, q, r be the perpendiculars from the vertices on the opposite sides a, b, c respectively, prove that sin. p sin. $a=$sin. q sin. $b=$sin. r sin. c

$$=2\sqrt{\text{sin. } s \text{ sin } (s-a) \text{ sin. } (s-b) \text{ sin. } (s-c)}.$$

8. The sides of a spherical triangle are all quadrants, and x, y, z the arcs joining any point within the triangle with the angular points, prove that

$$\cos.^2 x+\cos.^2 y+\cos.^2 z=1.$$

9. In a spherical triangle ABC, if the angle B be equal to the side c, show that

$$\text{sin. } (A-a)=\text{sin. } a \text{ sin. A cos. B cot. B}.$$

10. If the sum of two sides of a spherical triangle exceeds the third side by the semicircumference of a great circle, show that the sine of half the angle contained by these two sides is a mean proportional between their cotangents.

11. If in a spherical triangle ABC, the angle C is a right angle, prove that the following relations will hold :

(a) $\text{Cos. } (a+b)+\cos. (a-b)=2 \cos. c.$

(b) If tan. $a=2$ and tan. $b=1$, then tan. $c=3$.

(c) $\text{Tan.}^2 \frac{A}{2}=\frac{\text{sin. } (c-b)}{\text{sin. } c+b)}.$

(d) $\text{Tan.}^2 \frac{b}{2}=\tan. \frac{c+a}{2}\tan. \frac{c-a}{2}.$

(e) $\text{Sin. } (c+a) \text{ sin. } (c-a)=\sin.^2 b \cos.^2 a.$

(f) $\text{Sin.}^2 \frac{c}{2}=\sin.^2 \frac{a}{2} \cos.^2\frac{b}{2}+\cos.^2 \frac{a}{2}\sin.^2\frac{b}{2}.$

(*g*) $\text{Sin. }(c-a)=\tan. b \cos. c \tan. \frac{B}{2}=\sin. b \cos. a \tan. \frac{B}{2}$

(*h*) $\text{Sin. }(a-b)=\sin. a \tan. \frac{A}{2}-\sin. b \tan. \frac{B}{2}.$

(*i*) $\text{Sec.}^2\left(45^\circ+\frac{c}{2}\right)=\frac{2 \sin. B}{\sin. B-\sin. b}.$

12. In a spherical triangle ABC, if $b=c$, prove that

(*a*) $\cos. \frac{A}{2}=\cot. b \sin. \frac{a}{2} \tan. B.$

(*b*) $\sin.^2 B \cos.^2 \frac{a}{2} \sin.^2 b=\sin.^2 b-\sin.^2 \frac{a}{2}.$

13. In a spherical triangle ABC, having a right angle at C, if two arcs, x, y, be drawn from C to c, of which x is perpendicular to the side c, and y bisects it, prove that

(*a*) $\cot. x=\sqrt{\cot.^2 a+\cot.^2 b}.$

(*b*) $\cot. y=\frac{\cos. a+\cos. b}{\sqrt{\sin.^2 a+\sin.^2 b}}.$

(*c*) $\sin. \frac{c}{2}=\frac{\sin. y}{\sqrt{1+\sin.^2 x}}.$

14. Two great circles of a sphere PaA, PbB intersect two other great circles QBA, Qba in points A, B and a, b: prove that

$$\frac{\sin. AQ}{\sin. BQ}=\frac{\sin. Aa \sin. Pb}{\sin. Bb \sin. Pa}.$$

15. If E, F are the middle points of the sides AC, AB of a spherical triangle ABC, and EF produced meets BC produced at D, prove that

$$\sin. DE \cos. \frac{b}{2}=\sin. DF \cos. \frac{c}{2}.$$

16. The middle points of the sides AB, AC of a spherical triangle are joined by the arc of a great circle, which cuts the base produced towards C at D. Prove that $BD+CD=180^\circ$, and that

$$\cos. AD=\sin. \frac{b+c}{2} \sin. \frac{c-b}{2} \text{cosec.} \frac{a}{2},$$

AB being the greater of the two sides.

17. If D, E are the middle points of the sides AB, AC of an equilateral spherical triangle, prove that

$$\tan. \frac{BC}{2} = 2 \sin. \frac{DE}{2}.$$

18. If in the spherical triangle ABC, D is the middle point of BC, and AE is drawn perpendicular to BC, prove that

$$\tan. DE = \tan. \frac{b+c}{2} \tan. \frac{b-c}{2} \cot. \frac{a}{2}.$$

19. If in the spherical triangle ABC, AD is drawn perpendicular to BC, and DE perpendicular to AC, cutting AB in F, prove that

$$\frac{\tan. EF}{\tan. ED} = \cos. b \tan. A \tan. C.$$

20. In the spherical triangle ABC the angle $C = 120°$, show that the arc of a great circle drawn through C to meet AB at right angles is $\tan.^{-1}\frac{\sqrt{3}}{2}$, if

$$\cot.^2 a + \cot. a \cot. b + \cot.^2 b = 1.$$

21. Show that in any spherical triangle ABC,

$$\cos. A + \cos. B = \frac{2 \sin. (a+b)}{\sin. c} \sin.^2 \frac{C}{2}.$$

22. In a spherical triangle, if $b + c = 60°$, show that

$$\cos. (b - c) = \cos. a + \frac{\cos. \frac{3a}{2}}{2 \cos. \frac{a}{2}} \tan.^2 \frac{A}{2}.$$

23. If in a spherical triangle ABC, $a + b + c = 180°$, prove that

$$\sin.^2 \frac{A}{2} + \sin.^2 \frac{B}{2} + \sin.^2 \frac{C}{2} = 1.$$

24. If D is the middle point of AB, prove that

$$\cot. BCD - \cot. ACD = \frac{\sin.^2 A - \sin.^2 B}{\sin. A \sin. B \sin. C}.$$

25. If ABC be a spherical triangle, of which the sides are

each equal to a quadrant, and if R be the pole of the great circle passing through any two points P and Q, prove that

$$\cos. AR \sin. PQ = \cos. PB \cos. QC - \cos. PC \cos. QB.$$

26. If the sides a, b, c of a spherical triangle ABC are in arithmetical progression, show that

$$\cos. \frac{C-A}{2} = \frac{\sin.^2 \frac{B}{2}}{\sin. \frac{A}{2} \sin. \frac{C}{2}}.$$

27. If P be any point in the base AB of an isosceles spherical triangle ACB, prove that

$$\frac{\cos. \frac{1}{2}(AP-PB)}{\cos. \frac{1}{2} AB} = \frac{\cos. CP}{\cos. CA}.$$

28. If arcs be drawn from the angles of a spherical triangle to the middle points of the opposite sides, and if α, β be the parts of the arc which bisects the side a, show that

$$\frac{\sin. \alpha}{\sin. \beta} = 2 \cos. \frac{a}{2}.$$

29. If APB, DPE be two arcs of great circles meeting the same small circle of the sphere in A, B and D, E, and each other in P, then

$$\tan. \frac{AP}{2} \tan. \frac{BP}{2} = \tan. \frac{DP}{2} \tan. \frac{EP}{2}.$$

30. A spherical triangle ABC has the angle $C = 120°$, and $b = 3a$. Show that the length of an arc d bisecting C and terminated by the opposite side is given by the equation

$$\tan. d = \frac{1}{4} (\tan. a + \tan. 2a).$$

Part III.

PRACTICAL TRIGONOMETRY
PLANE AND SPHERICAL

CHAPTER I.

ON THE METHOD OF USING TABLES OF LOGARITHMS.

1. *To find from the tables the logarithm of a given number.* (See Part I., Chapters XIV. and XV.)

Example 1.—Find the logarithms of 7963 and ·02795.

Since 7963 is less than 10,000 the mantissa may be taken at once from the tables, and is found to be ·901077.

The characteristic is found by inspection to be 3 (Part I., art. 99).

Thus the complete logarithm is 3·901077.

To find the logarithm of the decimal fraction ·02795 we treat it as a whole number (Part I., art. 110) and look out the mantissa of 2795.

This is found to be ·446382.

The characteristic is seen to be $\overline{2}$.

Thus log. ·02795 is $\overline{2}$·446382.

Example 2.—Find the logarithm of 109872·5.

As before, we first treat 109872·5 as a whole number.

The mantissa of the logarithm of	1098000	is	·040602
The proportional parts for	700	are	278
,, ,, ,,	20	,,	079
,, ,, ,,	5	,,	198
	(sum)		·04088988

Rejecting the last two figures, and increasing the last figure retained by unity (Part I., art. 106), we have for the mantissa of the required logarithm ·040890; and the characteristic is 5. Thus

log. 109872·5 is 5·040890.

Example 3.—Find the logarithm of ·00384757.

Regarding the decimal given as a whole number,

The mantissa of 384700 is	·585122
The proportional parts for 50 are	056
" " " 7 "	079
(sum)	·5851859

and the appropriate characteristic is $\bar{3}$.

We have therefore

log. ·00384757 is $\bar{3}$·585186.

Should the number be given in the form of a vulgar fraction it may be converted into a decimal fraction, and its logarithm obtained as before.

Or the logarithm of the denominator may be taken from that of the numerator, and the result will be the logarithm of the given fraction. The method first given is, however, generally preferable.

Examples.—I.

Required the logarithms of the following numbers :—

1. 248.
2. 1476.
3. 14·06.
4. 4196·534.
5. 121004·15.
6. $49\frac{1}{4}$.
7. $\frac{1}{25}$.
8. $\frac{1}{2000}$.
9. ·75234.
10. $90\frac{1}{25}$.
11. $1000\frac{1}{125}$.
12. ·00732596.

2. *To find the number corresponding to a given logarithm.* (See Part I., arts. 111, 112.)

Example 1.—Let the given logarithm be 2·619406 ; it is required to find the corresponding natural number.

Turning to the tables we find that the given mantissa is that set down against the number 4163.

We know, therefore, from the characteristic given, viz. 2, that the number required is 416·3.

Example 2.—Let the given logarithm be 6·150822.

The mantissa given in the tables next below the value given is ·150756, corresponding to the number 1415. Thus we have :

The mantissa of the given logarithm . .	·150822			
The mantissa next lower in the tables . .	·150756	corresponding to 1415		
(difference)	66			
Prop. parts next lower . .	61	„	„	2
	50			
„ „ . .	31	„	„	1
	190			
„ „ . .	184	„	„	6
		(sum)		1415216

The process might be carried further if necessary.

Since the characteristic is 6, the natural number required will be 1415216 (Part I., art. 99).

Examples.—II.

Required the natural numbers corresponding to the following logarithms:—

1. 2·394452.	7. $\bar{2}$·380211.
2. 6·394452.	8. $\bar{5}$·544068.
3. 2·415671.	9. $\bar{1}$·841672.
4. ·217845.	10. $\bar{7}$·875061.
5. 2·310101.	11. $\bar{3}$·602060.
6. 5·082800.	12. $\bar{2}$·394452.

3. *To find the tabular logarithmic sine, cosine, &c. of an angle given to the nearest second.* (See Part I., arts. 115, 117.)

Example 1.—Let it be required to find the tabular logarithmic sine of 31° 11′ 12″.

We have from the tables

L sin. 31° 11′ 15″	9·714196
L sin. 31 11 0	9·714144
(difference) . .	·000052

Let x represent the excess of

L sin. 31° 11′ 12″ over L sin. 31° 11′ 0″.

Then x : ·000052 :: 12″ : 15″.

Therefore $x = \frac{4}{5}(\cdot 000052)$, or ·000042 nearly.

Hence L sin. 31° 11′ 12″ = 9·714144 + ·000042
= 9·714186.

If the angle given be in the second quadrant, since sin. (180° − A) = sin. A, cos. (180° − A) = − cos. A, &c. we must look out the logarithm of the same function of the supplement of the angle. Thus L sin. 137° = L sin. 43°.

Examples.—III.

1. Required the tabular logarithmic sines, tangents, and cosecants of the following angles:—

(1) 10° 10′ 6″. (3) 48° 35′ 35″.
(2) 19 10 40. (4) 61 24 40.

2. Find L hav. 61° 9′ 53″ and L hav. 135° 21′ 37″.
3. Find L sin. 102° 37′ 44″ and L hav. 215° 17′ 33″.

4. *To find to the nearest second the angle corresponding to a given tabular logarithmic sine, cosine, or haversine.* (See Part I., arts. 116, 118.)

Let it be required to find the angle which has for its tabular logarithmic sine 9·469003.

From the tables

L sin. 17° 7′ 30″ . . .	9·469022
L sin. 17 7 15 . . .	9·468920
(difference)	·000102
Tabular logarithm given	9·469003
L sin. 17° 7′ 15″	9·468920
(difference)	·000083

Let x represent the excess of the given angle over 17° 7′ 15″.

Then $x : 15'' :: \cdot000083 : \cdot000102$.

Thus $x = \frac{83}{102} 15'' = 12''$.

Therefore the required angle is 17° 7′ 27″.

Examples.—IV.

1. Find to the nearest second the value of the angle A in the following cases:—

(1) L sin. A = 9·641452. (3) L sec. A = 10·723465.
(2) L cot. A = 10·200970. (4) L tan. A = 9·488763.
(5). L hav. A = 9·757633.

2. What values of A, less than 360°, have the values given below for the several tabular logarithms?—

(1) L sin. A = 9·763520. (3) L tan. A = 10·790490.
(2) L cos. A = 9·840750. (4) L hav. A = 9·632720.

5. *To find from the tables the tabular versine of an angle, and to find the angle corresponding to a given tabular versine.* (See Part I., arts. 124, 125.)

Let it be required to find the tabular versine of 48° 29′ 47″.

We have from the tables,

Tabular versine of 48° 29′ . . .	0337162
Parts for 47″	169
Tabular versine of 48° 29′ 47″ . .	0337331

Example, 2.—Find the angle corresponding to the tabular versine of 0077895.

The given tabular versine is . . .	0077895
From the tables tab. vers. 22° 45′ is . .	0077799
(difference) . . .	96

Looking in the column of 'parts for seconds' headed by 22° 30′, we find that 96 corresponds to 52″.

The complete value of the angle required is therefore 22° 45′ 52″.

EXAMPLES.—V.

1. Find the tabular versines of the following angles :—

(1) 37° 56′ 42″. (2) 70° 18′ 10″. (3) 101° 27′ 32″.

2. Find the angles corresponding to the tabular versines

(1) 0072653. (2) 0987321. (3) 0003721.

Multiplication by logarithms.

6. By Part I., art. 90 the logarithm of a product is equal to the sum of the logarithms of the several factors; we have therefore only to take the logarithm of each number involved, and add these logarithms together. The sum will be the logarithm of the product.

EXAMPLES.—VI.

Find by logarithms the products of the following quantities :—

1. 84 × 96.
2. 6 × 4 × 12 × 32.
3. 36 × 48 × 62 × ·4.
4. 72 × 96 × 124 × ·05.
5. 2·4 × ·007 × ·54 × ·1.
6. 784 × ·000079 × ·0000036.

Division by logarithms.

7. Since, by Part I., art. 91, the logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor, we have only to write down the logarithm of the dividend and subtract from it the logarithm of the divisor. The difference will be the logarithm of the quotient.

Example 1.—Divide 472 by 32·2.

From the tables log. 472 . . .	2·673942
,, ,, log. 32·2 . . .	1·507856
(difference) log. quotient . .	1·166086

Whence the quotient is 14·658.

Example 2.—Divide ·0472 by 3·22.

log. ·0472 . . .	$\bar{2}$.673942
log. 3·22	·507856
(difference) log. quotient	$\bar{2}$·166086

Therefore the quotient is ·014658.

Examples.—VII.

1. Find by logarithms the value of

(1) 2004·64 divided by 34.
(2) 19 „ 72.
(3) 1 „ ·45.
(4) 1 „ ·0004572.

2. Find the value of the fractions

(1) $$\frac{242 \times 559 \times 63}{781 \times 432}.$$

(2) $$\frac{84 \times \cdot 00769 \times \cdot 683}{598 \times \cdot 0000146 \times \cdot 039}$$

Involution and Evolution by Logarithms.

8. By Part I., art. 92 the logarithm of any power, integral or fractional, of a number is equal to the logarithm of the number multiplied by the index of the power.

Example 1.—Required the value of $(1 \cdot 05)^{16}$.

From the tables	log. 1·05 . . .	·021189
		4
		·084756
		4
	log. $(1 \cdot 05)^{16}$	·339024

Thus $(1 \cdot 05)^{16} = 2 \cdot 18285$.

In the next example the quantity to be raised to a power is a decimal fraction, so that its logarithm will have the characteristic negative and the mantissa positive.

The multiplication of the two parts must therefore be

performed separately, and the required logarithm is the algebraical sum of the two products.

Example 2.—Find the value of $(\cdot 2)^{10}$.

$$\text{log.} \cdot 2 \quad . \quad . \quad . \quad \bar{1}\cdot 301030, \text{ or } \bar{1} + \cdot 301030$$
$$\underline{10} \qquad \underline{10}$$
$$\overline{10} + 3.010300$$

The algebraical sum of the two is $\bar{7}\cdot 010300$.

Thus $(\cdot 2)^{10} = \cdot 0000001024$.

If the index of the power be fractional, the process of division is substituted for that of multiplication.

Example 3.—Required the value of $(1234)^{\frac{1}{3}}$.

From the tables log. 1234 is 3·091315.

$$3\underline{)3\cdot 091315}$$
$$1\cdot 030438$$

Thus the value of $(1234)^{\frac{1}{3}}$ is 10·726.

Example 4.—Required the value of $(\cdot 00005214)^{\frac{1}{5}}$.

From the tables log. ·00005214 is $\bar{5}\cdot 717171$;

$$5\underline{)\bar{5}\cdot 717171}$$
$$\bar{1}\cdot 143434$$

Therefore $(\cdot 00005214)^{\frac{1}{5}} = \cdot 13913$.

A slight complication arises when the characteristic of the logarithm to be divided is negative, and does not contain the divisor an exact number of times. In this case it is customary to increase the negative characteristic by the least number of units which will render it divisible by the divisor without remainder, adding to the positive portion of the logarithm, that is, to the mantissa, the same number of units, so that the value of the whole logarithm remains unaltered.

Thus to divide $\bar{4}\cdot 681241$ by 3. By adding and subtracting 2 we may write the logarithm in the form

$\bar{6} + 2\cdot 681241$, which when divided by 3 is $\bar{2}\cdot 893747$.

Example 5.—Find the value of $(\cdot 00345)^{\frac{1}{10}}$.

Log. ·00345 is $\bar{3}\cdot 537819$, or $\overline{10} + 7\cdot 537819$.

Therefore log. $(\cdot 00345)^{\frac{1}{10}}$ is $\bar{1}\cdot 753782$,

and the value of $(\cdot 00345)^{\frac{1}{10}}$ is ·56726.

If the index of the power be negative it will in general be easiest to reduce it in the first place to an equivalent expression with a positive index, by substituting for the given quantity its reciprocal with the sign changed.

$$\text{Thus } a^{-n} = \frac{1}{a^n};\ x^{-3} = \frac{1}{x^3};\ \frac{a^{-n}}{b^{-p}} = \frac{b^p}{a^n},$$

and so on.

Example 6.—Find the value of $(\cdot4)^{-5}$.

We have by algebra

$$(\cdot4)^{-5} = \left(\frac{1}{\cdot4}\right)^5$$

$$\text{Now log. } \left(\frac{1}{\cdot4}\right)^5 = \text{log. } \frac{1}{(\cdot4)^5} = \text{log. } 1 - 5 \text{ log. } \cdot4.$$

From the tables

log. 1 .	·000000	log. ·4 . .	$\bar{1}\cdot602060$
5 log. ·4 .	$\bar{2}\cdot010300$		5
(difference)	$1\cdot989700$	5 log. ·4 .	$\bar{2}\cdot010300$

Thus the value of $(\cdot4)^{-5} = 97\cdot656$.

Examples.—VIII.

Find by logarithms the values of the following quantities :—

1. $(12\cdot5)^3$.
2. $(4\cdot7215)^6$.
3. $(1\cdot05)^{150}$.
4. $(1\cdot0125)^{200}$.
5. $(1\cdot0125)^{1000}$.
6. $(\cdot2)^5$.
7. $(\cdot09163)^4$.
8. $(\cdot975)^{200}$.
9. $(784)^{\frac{1}{5}}$.
10. $\sqrt{365}$.
11. $\sqrt[3]{12345}$.
12. $(2)^{\frac{1}{10}}$.
13. $\sqrt{\cdot093}$.
14. $(7\cdot0825)^{\frac{1}{5}}$.
15. $(\cdot00125)^{\frac{1}{3}}$.
16. $(\cdot0125)^{\frac{1}{3}}$.
17. $\sqrt{\cdot0093}$.
18. $(\cdot000048)^{\frac{1}{3}}$.
19. $(19)^{\frac{4}{5}}$.
20. $(\cdot096)^{\frac{5}{9}}$.
21. $(472)^{\frac{5}{9}}$.
22. $\dfrac{(466871)^{\frac{6}{7}} \times \sqrt[9]{(3576)^{16}}}{996003 \times \sqrt{\cdot0077}}$.
23. $(\cdot042)^{8\cdot3}$.
24. $(\cdot00563)^{\cdot07}$.
25. 3^{-7}.
26. $3^{-\frac{1}{7}}$.
27. $(\cdot045)^{-\frac{1}{5}}$.
28. $\dfrac{1}{(\cdot2)^{-4}}$.

9. *To adapt an expression to logarithmic computation.*

Let it be required to find by logarithms the value of x in the equation

$$\sqrt{x} = \frac{\sqrt{a}\, b\, \sqrt[4]{c}}{d^2}.$$

By making use of the properties of logarithms established in Part I., Chapter XIV., we may adapt the equation to logarithmic computation as follows:—

$$\frac{1}{2}\log. x = \frac{1}{2}\log. a + \log. b + \frac{1}{4}\log. c - 2\log. d.$$

Examples.—IX.

1. Adapt the following equations to logarithmic calculation:—

(1) $x = a^2 b c d^2$.

(2) $x = \dfrac{a^2 b \sqrt[3]{c}}{10}$.

(3) $x = \dfrac{\sqrt{a b}}{c d^2}$.

(4) $x = \dfrac{\sqrt[3]{a b c^4}}{\sqrt[4]{d}}$.

2. Find the value of the following expressions:—

(1) $\dfrac{3^7}{9^2 \times 4^{-6}}$.

(2) $\dfrac{3^7 \times 4^{-9}}{3^{-2}}$.

(3) $\dfrac{4^{-5} \times 5^{-\frac{2}{3}}}{35 \times \sqrt{2}}$.

(4) $\dfrac{7\sqrt{15}}{\cdot 015} \times \cdot 0139 \sqrt{\dfrac{2}{\cdot 11}}$.

(5) $(6)^{\sqrt{5}}$.

(6) $\dfrac{2\cdot\left(\dfrac{3}{2}\right)^{15} - 2}{\dfrac{3}{2} - 1}$.

(7) $\left\{\dfrac{\dfrac{5}{9}\sqrt[18]{2}}{\dfrac{10}{17}\sqrt[23]{3}} + (5)^{\frac{4}{5}}\right\}^3$.

3. Find an approximate value of x in the following equations:—

(1) $x^3 = 14$.

(2) $3^{2x} = 20$.

(3) $x^3 = \cdot 004$.

(4) $x^{-3} = 4\frac{1}{2}$.

(5) $x = (\cdot 02445)^{\frac{3}{4}}$.

(6) $x = \sqrt[2000]{47691}$.

(7) $x = \sqrt{\dfrac{2}{123}}$.

(8) $x = \sqrt[3]{\dfrac{1}{3\cdot 14159}}$.

4. Show how x may be determined by means of logarithms in the equations

$$(1)\ a^x = b. \qquad (2)\ \frac{a^{mx}}{b^{nx-1}} = c. \qquad (3)\ a^x = \frac{b^{mx-n}}{c^{rx}}.$$

5. Find x in the equations

$$(1)\ 4^{3x} = \cdot 005. \qquad (2)\ (\cdot 04)^{3x} = \cdot 001.$$

$$(3)\ (\cdot 04)^{\sqrt{x}} = 5.$$

10. *The expression of a trigonometrical formula in logarithms.*

In the reduction of an expression involving the trigonometrical ratios to logarithms, it must be remembered that the logarithms of the several ratios given in the tables are increased by 10. In the final result, therefore, due allowance must be made for the several tens belonging to the tabular logarithms employed.

Attention must also be given to the algebraic signs of the different ratios.

Example 1.—Find x in the equation

$$\frac{7\cdot 8 \text{ sin. } 49^\circ \text{ cosec. } 112^\circ}{x^2} = \text{tan. } 52^\circ.$$

Multiplying both sides by x^2, and transposing, we have

$$x^2 \text{ tan. } 52^\circ = 7\cdot 8 \text{ sin. } 49^\circ \text{ cosec } 112^\circ,$$

$$\therefore\ x^2 = 7\cdot 8 \overset{+}{\text{ sin. }} 49^\circ \overset{+}{\text{ cosec. }} 68^\circ \overset{+}{\text{ cot. }} 52^\circ;$$

that is, $2 \log. x = \log. 7\cdot 8 + \text{L sin.} 49^\circ + \text{L cosec. } 68^\circ + \text{L cot. } 52^\circ - 30.$

	log. 7·8	·892095
	L sin. 49°	9·877780
	L cosec. 68° . . .	10·032834
	L cot. 52°	9·892810
	(sum)	30·695519
		30·000000
(difference)	2 log. x . .	·695519

Therefore $\log. x = \frac{1}{2} (\cdot 695519) = \cdot 347759$, and $x = 2\cdot 227$ nearly.

Example 2.—Find x in the equation

$$\frac{120\cdot5 \cos. 130^\circ}{x} = 73 \sec. 70^\circ \tan.^2 210^\circ.$$

$$\text{Therefore } 73\,x = \overset{-}{120\cdot5 \cos. 130^\circ} \overset{-}{\cos. 70^\circ} \overset{+}{\cot.^2} \overset{+}{210^\circ};$$

$$\log. x = \log. 120\cdot5 + L \cos. 130^\circ + L \cos. 70^\circ + 2\,L \cot. 210 - 40 - \log. 73.$$

log. 120·5 . . .	2·080987
L cos. 130°	9·808067
L cos. 70°	9·534052
2 L cot. 210°	20·477122
	41·900228
	40·000000
(difference). .	1·900228
log. 73 . .	1·863323
(difference). .	·036905

In this case, as appears from the equation, since cos. 130° is negative, the value of x must be negative also.

Therefore $x = -1\cdot089$ nearly.

Examples.—X.

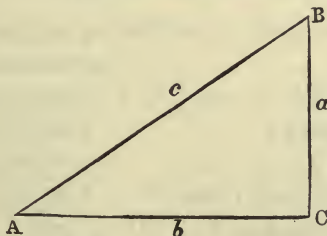
Find x in the equations

1. $x \sin. 68^\circ = 117 \tan. 48^\circ \sec.^2 10^\circ.$
2. $\dfrac{\text{cosec. } 100^\circ}{\sqrt{x}} = \dfrac{8\cdot5 \sin.^2 50^\circ}{\sqrt{\tan. 61^\circ}}.$
3. $x^3 \cot. 109^\circ = 129\cdot6 \sin. 73^\circ \text{ cosec. } 119^\circ.$
4. $\dfrac{41\cdot3 \tan. 200^\circ}{\sqrt[3]{x}} = \dfrac{\sqrt[3]{\sin. 80^\circ}}{(\cdot75)^{-2}}.$

CHAPTER II.

THE SOLUTION OF RIGHT-ANGLED PLANE TRIANGLES.

(See Part I., Chapter XVII.)

11. *Example* 1.—In the plane triangle ABC let $a = 117$, $b = 201$, and $C = 90°$, required the other parts.

(1) *To find the angle* A.

Since $\tan. A = \frac{a}{b}$, $L \tan. A = 10 + \log. a - \log. b$.

	10·000000
log. 117	2·068186
(sum)	12·068186
log. 201	2·303196
(difference) L tan. A.	9·764990

Therefore $A = 30° \, 12' \, 15''$.

(2) *To find the angle* B.

Since $A + B = 90°$, $B = 90° - 30° \, 12' \, 15'' = 59° \, 47' \, 45''$.

(3) *To find the side c.*

Since $c = a \text{ cosec. } A$,

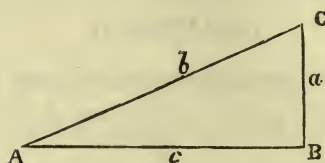
$\log. c = \log. a + L \text{ cosec. } A - 10$.

log. 117.	2·068186
L cosec. 30° 12′ 15″. .	10·298361
(sum)	12·366547
	10·000000
(difference)	2·366547

Therefore $c = 232·5$.

Example 2.—In the plane triangle ABC, having given $a = \cdot 02$, $c = \cdot 1$, and $B = 90°$, find the other parts.

(1) *To find the angle* A.

$$\text{Since tan. } A = \frac{a}{c},$$

therefore L tan. A = 10 + log. a − log. c.

	10·000000
log. ·02	$\bar{2}$·301030
(sum)	8·301030
log. ·1	$\bar{1}$·000000
(difference) L tan. A.	9·301030

Therefore A = 11° 18′ 30″.

(2) *To find the angle* C.

Since A + C = 90°, C = 90° − 11° 18′ 30″ = 78° 41′ 30″.

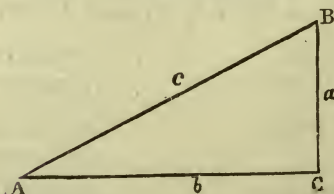
(3) *To find the side b.*

Since $b = c$ sec. A,

log. b = log. c + L sec. A − 10.

log. ·1	$\bar{1}$·000000
L sec. 11° 18′ 30″ . .	10·008514
(sum)	9·008514
	10·000000
(difference) log. b. .	$\bar{1}$·008514

Therefore b = ·102.

Example 3.—In the plane triangle ABC, having given $b = 33$, A = 37° 40′, and C = 90°, required the other parts.

(1) *To find the angle* B.

Since A + B = 90°, B = 90° − 37° 40′ = 52° 20′.

(2) *To find the side a.*

Since $a = b \tan. A$,

$\therefore \log. a = \log. b + L \tan. A - 10.$

log. 33	1·518514
L tan. 37° 40′ . . .	9·887594
(sum)	11·406108
	10·000000
(difference) log. a . .	1·406113

Therefore $a = 25·5$.

(3) *To find the side c.*

Since $c = b \sec. A$,

$\log. c = \log. b + L \sec. A - 10.$

log. 33	1·518514
L sec. 37° 40′ . . .	10·101506
(sum)	11·620020
	10·000000
(difference) log. c . .	1·620020

and $c = 41·7$.

EXAMPLES.—XI.

1. In the plane triangle ABC find the other parts, having given

(1) $a = 35·76$, $b = 45$, $B = 90°$.
(2) $a = 384$, $c = 331$, $B = 90°$.
(3) $a = 3555$, $b = 2354$, $C = 90°$.
(4) $b = ·2$ $A = 40°$, $B = 90°$.
(5) $c = ·04$, $C = 40°$, $B = 90°$.
(6) $b = 1777·5$, $c = 1177$, $A = 90°$.

2. The sides of a rectangular field are 156 feet and 117 feet respectively; find the distance between two opposite corners.

3. At a point 153 feet from the foot of a tower the angle of elevation (Part I., art. 164) of the top of the tower is 57° 19′; find the height of the tower.

4. From the top of a cliff, known to be 100 feet above the sea-level, the angle of depression (Part I., art. 164) of a boat at sea is 11° 19′. How far is the boat from the foot of the cliff?

5. A ladder 52 feet in length reaches a window 37 feet from the ground. When the ladder is turned over about its foot

it reaches a window-sill on the other side of the street 43 feet from the ground. Find the width of the street.

6. The length of the shadow of a perpendicular stick is 6·25 feet, when the altitude of the sun is 31° 17′. What will be the length of the shadow of the same stick when the altitude of the sun is 50° 26′?

7. A lighthouse bore N.N.E. from a ship at anchor. After the ship had sailed E.N.E. 17 miles the lighthouse bore N.N.W. What was the distance of the anchorage from the lighthouse?

8. A column 100 feet high, standing in the middle of a square court, subtended an angle of 21° at a corner of the court. Find the length of a side of the court.

9. A certain port, B, is due south of another port, A, distant 40 miles. At noon a ship left A, steaming S. x° W., 3 knots; and at the same time a second ship left B, steaming N. $(90 - x)$° W., 4 knots. The two ships met the same evening. At what time did this occur?

CHAPTER III.

THE SOLUTION OF OBLIQUE-ANGLED PLANE TRIANGLES.

CASE I.

12. *Having given the three sides of a plane triangle, to find the angles.* (See Part. I., arts. 152–154.)

In the plane triangle ABC having given $a = 20$, $b = 30$, $c = 40$: find the angles A, B, C.

(1) *To find the angle* A.

Since $$\text{hav. A} = \frac{(s-b)(s-c)}{bc},$$

therefore $\text{L hav. A} = 10 + \log.(s-b) + \log.(s-c) - \log. b - \log. c$.

a . .	20	s . .	45	s . .	45
b . .	30	b . .	30	c . .	40
c . .	40	$s-b$.	15	$s-c$.	5
(sum)	90				

$\frac{1}{2}$ (sum) $= s = 45$.

	10·000000		
log. 15	1·176091	log. 30 . .	1·477121
log. 5	·698970	log. 40 . .	1·602060
(sum)	11·875061	(sum)	3·079181
	3·079181		
(difference) L hav. A	8·795880		

$\therefore$ A = 28° 57′ 15″.

(2) *To find the angle* B.

As before,

$$\text{L hav. B} = 10 + \log.(s-a) + \log.(s-c) - \log. a - \log. c.$$

s . . .	45	s . . .	45
a . . .	20	c . . .	40
$s-a$ . .	25	$s-c$ . .	5

	10·000000		
log. 25	1·397940	log. 20 . .	1·301030
log. 5	·698970	log. 40 . .	1·602060
(sum)	12·096910	(sum)	2·903090
	2·903090		
(difference) L hav. B	9·193820		

$\therefore$ B = 46° 34′ 0″.

(3) *To find the angle* C.

The angle C may now be obtained by subtracting the sum of A and B from 180°.

Thus C = 180° − 28° 57′ 15″ − 46° 34′ 0″ = 104° 28′ 45″.

Or C may be obtained by a third application of the haversine formula.

Thus

$$\text{L hav. C} = 10 + \log.(s-a) + \log.(s-b) - \log. a - \log. b$$

s . . .	45	s . . .	45
a . . .	20	b . . .	30
$s-a$ . .	25	$s-b$ . .	15

	10·000000		
log. 25	1·397940	log. 20 . .	1·301030
log. 15	1·176091	log. 30 . .	1·477121
(sum)	12·574031	(sum) . .	2·778151
	2·778151		
(difference) L hav. C	9·795880		

$\therefore C = 104° 28' 45''$.

Note.—In finding the second angle B, we may, if we please, make use of the relation

$$\sin. B = \frac{b}{a} \sin. A.$$

In that case, however, an error made in calculating the angle A will affect the result of B also. It is preferable, therefore, at the expense of slightly increased labour in computation, to obtain B directly from the original data.

Examples.—XII.

In the plane triangle ABC find the angles A, B, C, having given

1. $a = 798$, $b = 460$, $c = 654$.
2. $a = 512$, $b = 627$, $c = 430$.
3. $a = 649$, $b = 586$, $c = 757$.
4. $a = 627$, $b = 1140$, $c = 718·9$.
5. $a = ·025$, $b = ·125$, $c = ·115$.
6. $a = ·8$, $b = ·672$, $c = ·275$.
7. $a = ·5$, $b = ·75$, $c = 1·013$.
8. $a = ·25$, $b = ·541$, $c = ·674$.

Case II.

13. *In a plane triangle, having given two angles and one side, to find the other parts.* (See Part. I., arts. 155, 156.)

In the plane triangle ABC, having given $A = 70° 36'$, $B = 57° 19'$, and $c = 37$, required the other parts.

(1) *To find the angle* C.

Since A + B + C = 180°, C = 180° − 70° 36′ − 57° 19′ = 52° 5′.

(2) *To find the side a.*

Since $\frac{a}{c} = \frac{\text{sin. A}}{\text{sin. C}}$, log. a = log. c + L sin A + L cosec. C − 20.

log. 37	1·568202
L sin. 70° 36′ . . .	9·974614
L cosec. 52° 5′ . .	10·102975
(sum)	21·645791
	20·000000
(difference) log. a	1·645791

∴ a = 45·2.

(3) *To find the side b.*

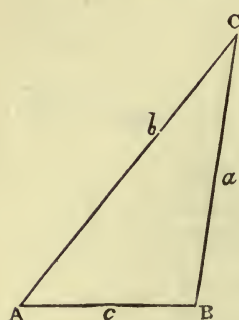
Since $\frac{b}{c} = \frac{\text{sin. B}}{\text{sin. C}}$, log. b = log. c + L sin. B + L cosec. C − 20.

log. 37	1·568202
L sin. 57° 19′ . . .	9·925141
L cosec. 52° 5′ . .	10·102975
(sum)	21·596318
	20·000000
(difference) log. b	1·596318

∴ b = 39·5.

Case III.

14. *In a plane triangle, having given two sides and the angle opposite to one of them, to find the other parts.* (See Part I., arts. 157–160).

Example 1.—In the plane triangle ABC, having given b = 56, c = 38, and B = 101° 20′, required the other parts.

(1) *To find the angle* C.

Since $\frac{\sin. C}{\sin. B} = \frac{c}{b}$,

therefore

$$L \sin. C = L \sin. B + \log. c - \log. b$$

L sin. 101° 20′ . . .	9·991448
log. 38	1·579784
(sum)	11·571232
log. 56	1·748188
(difference) L sin C . .	9·823044

$\therefore$ C $= 41° \, 42' \, 30''$.

(2) *To find the angle* A.

As before,

$$A = 180° - 101° \, 20' - 41° \, 42' \, 30'' = 36° \, 57' \, 30''.$$

(3) *To find the side a.*

Since $\frac{a}{b} = \frac{\sin A}{\sin B}$,

therefore $\log. a = \log. b + L \sin. A + L \operatorname{cosec.} B - 20.$

log. 56	1·748188
L sin. 36° 57′ 30″ .	9·779044
L cosec. 101° 20′ . .	10·008552
(sum)	21·535784
	20·000000
(difference) log. a . .	1·535734

$\therefore a = 34·3.$

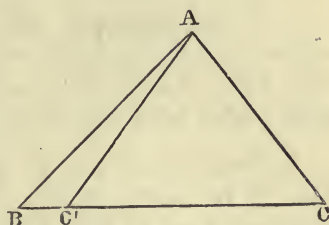
In the preceding example, since the angle given was that opposite to the *greater* side, the parts which remained to be found were determined without ambiguity.

The example which follows will illustrate the ambiguity discussed in Part. I., arts. 158–160.

Example 2.—In the plane triangle ABC, having given $b = 73$, $c = 101$, and $B = 39° 33'$, required the angle C.

$$\text{L sin. C} = \text{L sin. B} + \log. c - \log. b.$$

L sin. 39° 33′ . .	9·803970
log 101	2·004321
(sum)	11·808291
log. 73	1·863323
(difference) L sin. C	9·944968

∴ $C = 61° 45' 45''$, as in the triangle ACB; or $118° 14' 15''$, as in the triangle AC′B.

The remaining angles A, A′ and the remaining sides a, a' of the triangles ABC, A′BC may be found as in Example 1.

EXAMPLES.—XIII.

1. In the plane triangle ABC, find the other parts, having given

(1) $a = 214$, $b = 191$, $A = 41° 19' 15''$.
(2) $a = 17{\cdot}25$, $c = 10{\cdot}75$, $A = 47° \ 0' 30''$.
(3) $a = 96$, $c = 48$, $A = 101° 41'$.
(4) $a = 2{\cdot}75$, $A = 43° 24' 15''$, $B = 48° 33' 15''$.
(5) $a = {\cdot}5$, $b = {\cdot}75$, $B = 45° 10'$.
(6) $c = 376$, $A = 48° 3'$, $B = 40° 14'$.
(7) $a = 242$, $A = 60°$, $B = 72°$.
(8) $a = 178{\cdot}3$, $b = 145$, $B = 41° 10'$.
(9) $a = 2597{\cdot}84$, $b = 3084{\cdot}33$, $A = 56° 12' 45''$.

2. A fort bears from a ship at anchor S. 75° E., and is 2·35 miles distant; and a lighthouse bears S. 5° W. from the same ship. If the distance of the fort from the lighthouse be 3·25 miles, find the bearing of the one from the other.

3. From a ship steering W. by S. a beacon bore N.N.W., and after the ship had sailed 12 miles farther the bearing of the beacon was N.E. by E.: at what distance had the ship passed the beacon?

4. A boat is sent out from a ship with orders to row S. by E. until a rock situated 12 miles E.S.E. of the ship bears N.E. by N. How long will the boat take to reach the required position, rowing at the rate of 5 miles an hour?

5. A ship which can sail within seven points of the wind wishes to reach the mouth of a river 25 miles N.E. of her. The wind being N.N.E. she starts on the port tack, sailing 7·5 knots: after what interval should she go about?

6. Two forts, A and B, were four miles apart, and A bore from B, W. by N. A ship ordered to run in with A bearing due north until B bore E.N.E., anchored by mistake when B bore N.E.: how far was she then from her required position?

Case IV.

15. *In a plane triangle, having given two sides and the included angle, to find the other parts.* (See Part I., art. 161.)

In the plane triangle ABC let $a = 798$, $b = 460$, and C $= 55° \; 2' \; 15''$, it is required to find the remaining parts A, B, and c.

(1) *To find the angles* A *and* B.

By Part I, art. 140,

$$\tan. \frac{A-B}{2} = \frac{a-b}{a+b} \cot. \frac{C}{2};$$

therefore $L \tan. \dfrac{A-B}{2} = \log. (a-b) + L \cot. \dfrac{C}{2} - \log. (a+b)$.

a . .	798	A+B+C. .	180° 0′ 0″
b . .	460	C. .	55 2 30
$a+b$.	1258	(difference) A+B	124 57 30
$a-b$.	338	$\frac{A+B}{2}$ . . .	62 28 45
		$\therefore \frac{C}{2} =$	27 31 15

log. 338	2·528917
L cot. 27° 31′ 15″ . .	10·283138
(sum)	12·812055
log. 1258	3·099681
(difference) L tan. $\frac{A-B}{2}$	9·712374

$$\therefore \quad \frac{A-B}{2} = 27^\circ\ 16'\ 45''$$

$$\text{and} \quad \frac{A+B}{2} = 62\quad 28\quad 45$$

$$(\text{sum})\ A = 89\quad 45\quad 30$$

$$(\text{difference})\ B = 35\quad 12\quad 0$$

(2) *To find the side c.*

$$\text{Since} \quad \frac{c}{a} = \frac{\sin. C}{\sin. A},$$

therefore log. c = log. a + L sin. C + L cosec. A − 20.

log. a	2·902003
L sin. C	9·913563
L cosec. A	10·000004
(sum)	22·815570
	20·000000
(difference) log. c . .	2·815570

$$\therefore c = 654.$$

Examples.—XIV.

1. In the plane triangle ABC find the other parts, having given

(1) $b = 64$, $c = 70$, $A = 66^\circ\ 20'\ 30''$.
(2) $a = 512$, $b = 627$, $C = 42^\circ\ 53'\ 45''$.
(3) $b = 54$, $c = 79$, $A = 105^\circ\ 27'\ 30''$.
(4) $a = \cdot 036$, $b = \cdot 027$, $C = 75^\circ\ 16'\ 30''$.

2. Two ships leave a harbour at noon; one sails N.E. by N., 7 knots, the other E. by S., 9 knots. How far will they be apart at midnight?

3. From a certain station a fort, A, bore N., and a second fort, B, N.E. by E. Guns are fired simultaneously from the two forts, and are heard at the station in 1·5 seconds and 2 seconds respectively. Assuming that sound travels at the rate of 1,100 feet per second, find the distance of the two forts apart.

4. A boat is anchored half a mile from one end of a breakwater, and three-quarters of a mile from the other end, and an observer with a sextant finds that the breakwater subtends an angle of 50°. Find the length of the breakwater.

5. A ship was moored 7 miles N.N.E. of a lighthouse, and a boat, breaking adrift from the ship, was picked up four hours after 4 miles due east of the lighthouse. Find the direction and rate of the current by which the boat was set.

CHAPTER IV.

AREAS OF PLANE TRIANGLES.

Case I.

16. *In a plane triangle, having given two sides and the included angle, to find the area.* (See Part I., Art. 162.)

In the plane triangle ABC let $a = 798$ feet, $b = 460$ feet, and $C = 55° 2' 15''$, it is required to find the area.

$$\text{Since area} = \frac{1}{2} ab \sin. C,$$

therefore log. area = log. a + log. b + L sin. C − log. 2 − 10.

log. 798 . . .	2·902003	log. 2 ·	·301030
log. 460 . . .	2·662758		10·000000
L sin. C	9·913563	(sum)	10·301030
(sum)	15·478324		
	10·301030		
(difference) log. area	5·177294		

Therefore area = 150416 square feet nearly.

The work may be slightly simplified by dividing one of the sides by 2 in the first instance.

Thus,

$$\text{area} = \frac{1}{2}\,798 \times 460 \text{ sin. } 55^\circ\, 2'\, 15''.$$

or,

$$= 399 \times 460 \text{ sin. } 55^\circ\, 2'\, 15''.$$

So that

$$\text{log. area} = \text{log. } 399 + \text{log. } 460 + \text{L sin. } 55^\circ\, 2'\, 15'' - 10.$$

EXAMPLES.—XV.

1. In the plane triangle ABC find the area, having given

(1) $a = 245$ yards, $b = 760$ yards, $C = 60^\circ$.
(2) $b = 53$ feet, $c = 91$ feet, $A = 71^\circ\, 36'\, 30''$.
(3) $a = 78$ feet, $b = 101$ feet, $C = 109\; 27'\, 30''$.
(4) $a = 1{\cdot}23$ feet, $b = {\cdot}97$ feet, $C = 81^\circ\, 40'\, 0''$.
(5) $a = 103$ feet, $c = 76$ feet, $A = 95^\circ\, 37'\, 15''$.

2. An enclosure has the form of a parallelogram, the sides of which are $1\frac{1}{4}$ miles and 7 furlongs respectively, and the smaller of the angles included by the sides is 50°. Calculate its area in acres.

3. An isosceles triangle, the vertical angle of which is 78°, contains 100 square yards. Find the lengths of the equal sides of the triangle.

4. The area of a triangle is 763 square feet, and the sides which contain the smallest angle are 51 feet and 65 feet respectively. Find the angles of the triangle.

5. The base of an isosceles triangle is 150 feet, and the vertical angle 100°. Find the side of the equilateral triangle which has an area equal to that of the given triangle.

CASE II.

17. *Three sides of a plane triangle being given, to find the area.* (See Part I. art. 163.)

In the plane triangle ABC let $a = 711$ feet, $b = 681$ feet, $c = 327$ feet, to find the area.

By Part I., art. 142

$$\text{area} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

Therefore

$$\log. \text{area} = \tfrac{1}{2}\{\log. s + \log. (s-a) + \log. (s-b) + \log. (s-c)\}.$$

$a = 711$	s . 859·5	s . 859·5	s . 859·5
$b = 681$	a . 711·0	b . 681·0	c . 327·0
$c = 327$	$s-a$. 148·5	$s-b$. 178·5	$s-c$. 532·5
1719			
$s = 859·5$			

log. 859·5	2·934246
log. 148·5	2·171726
log. 178·5	2·251638
log. 532·5	2·726320
(sum)	2)10·083930
	5·041965

Therefore area = 110145 square feet.

EXAMPLES.—XVI.

1. In the plane triangle ABC find the area, having given

(1) $a = 101·4$ feet, $b = 76·5$ feet, $c = 91·3$ feet.
(2) $a = 1721$ feet, $b = 1946$ feet, $c = 2030$ feet,

2. Having given $a = ·08$ feet, $b = ·12$ feet, $c = ·15$ feet; find the number of square inches which the triangle contains.

3. Express in acres the area of a triangular field, the sides of which are 125 yards, 143 yards, and 159 yards respectively.

4. In a quadrilateral figure ABCD, AB = 90 yards, BC = 100 yards, CD = 110 yards, DA = 120 yards, and BD = 178·8 yards. Find the area of the figure.

CHAPTER V.

THE SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

Case I.

18. *Three sides of a spherical triangle being given, to find the angles.* (See Part II., art. 49.)

In the spherical triangle ABC let $a = 124° \ 10'$, $b = 89° \ 0' \ 15''$, $c = 108° \ 40'$, it is required to find the angle A.

To find the angle A.

By Part II., art. 43

$$\text{hav. A} = \text{cosec. } b \text{ cosec. } c \sqrt{\text{hav. } (a + \overline{b \sim c}) \text{ hav. } (a - \overline{b \sim c})}.$$

It has been explained in Part. II. that by taking the logarithms of cosec. b and cosec. c, instead of the tabular logarithms, we avoid the necessity of subtracting 20 in the final result.

Therefore

$$\text{L hav. A} = \text{log. cosec. } b + \text{log. cosec. } c + \tfrac{1}{2} \text{ L hav. } (a + \overline{b \sim c})$$
$$+ \tfrac{1}{2} \text{ L hav. } (a - \overline{b \sim c}).$$

c . .	108° 40′ 0″	log. cosec. c . .	·023468
b . .	89 0 15	log. cosec. b . .	·000066
$c - b$ . .	19 39 45	$\frac{1}{2}$ L hav. $(a + \overline{c - b})$.	4·978001
a . .	124 10 0	$\frac{1}{2}$ L hav. $(a - \overline{c - b})$.	4·898006
$a + \overline{c - b}$.	143 49 45	(sum) L hav. A .	9·899541
$a - \overline{c - b}$.	104 30 15		

Therefore A = 125° 56′ 30″.

By the same process we may find the angles B and C from the formulæ.

$$\text{L hav. B} = \text{log. cosec. } a + \text{log. cosec. } c + \tfrac{1}{2} \text{ L hav. } (b + \overline{a \sim c})$$
$$+ \tfrac{1}{2} \text{ L hav. } (b - \overline{a \sim c}).$$

$$\text{L hav. C} = \text{log. cosec. } a + \text{log. cosec. } b + \tfrac{1}{2} \text{ L hav. } (c + \overline{a \sim b})$$
$$+ \tfrac{1}{2} \text{ L hav. } (c - \overline{a \sim b}).$$

Note.—The formula

$$\sin. B = \frac{\sin. b}{\sin. a} \sin. A$$

might be used in finding B. Its use, however, is open to two objections: the first, that it involves the angle A, which is not one of the original data of the question; the second, that, as the angle is determined from its sine, we shall have to consider whether the angle first taken from the tables, or the supplement of that angle, is the value required, whereas by the haversine formula B is determined without ambiguity. In this case, therefore, as in similar instances which will be met with later, it is recommended that the use of the sine formula should in general be avoided.

EXAMPLES.—XVII.

In the spherical triangle ABC find the angles, having given

1. $a = 49° 10' 0''$, $b = 58° 25' 0''$, $c = 56° 42' 0''$.
2. $a = 119° 42' 30''$, $b = 108° 4' 15''$, $c = 68° 53' 45''$.
3. $a = 87° 10' 15''$, $b = 62° 36' 45''$, $c = 100° 10' 15''$.
4. $a = 156° 10' 30''$, $b = 137° 3' 45''$, $c = 47° 57' 15''$.
5. $a = 40° 31' 15''$, $b = 50° 30' 30''$, $c = 61° 5' 0''$.

CASE II.

19. *Having given two sides of a spherical triangle, and the angle included by those sides, to find the other parts.* (See Part II., art. 50.)

In the spherical triangle ABC, having given $b = 108° 4' 15''$, $c = 119° 42' 15''$, and $A = 75° 31' 30''$, it is required to find the other parts.

To find the side a

Since vers. a = vers. $(b \sim c)$ + sin. b sin. c. vers. A,

tab. vers. a = tab. vers. $(b \sim c)$ + tab. vers. θ, where

L hav. θ = L sin. b + L sin. c + L hav. A − 20.

To find θ.

L sin. b . . .	9·978031	c .	119° 42′ 15″
L sin. c . . .	9·938818	b .	108 4 15
L hav. A. . .	9·574056		11 38 0
(sum)	29·490905		
	20·000000		
(difference) L hav. θ	9·490905		

and $\theta = 67° \, 37' \, 30''$.

tab. vers. 67° 37′ . . .	619199
parts for 30″	134
tab. vers. 11° 38′ . . .	20542
(sum) tab. vers. a	639875

Therefore $a = 68° \, 53' \, 32''$.

Having now the three sides a, b, c, we may find the angles B, C by the method given under case I.

Or the angles B and C may be found directly from the data by means of Napier's Analogies. (Part II., art. 51.)

Since by Part II., art. 47

$$\tan. \tfrac{1}{2}(C+B) = \frac{\cos. \tfrac{1}{2}(c-b)}{\cos. \tfrac{1}{2}(c+b)} \cot. \frac{A}{2}.$$

$$\tan. \tfrac{1}{2}(C-B) = \frac{\sin. \tfrac{1}{2}(c-b)}{\sin. \tfrac{1}{2}(c+b)} \cot. \frac{A}{2}.$$

therefore

$$L \tan. \tfrac{1}{2}(C+B) = L \cos. \tfrac{1}{2}(c-b) + L \sec. \tfrac{1}{2}(c+b) + L \cot. \frac{A}{2} - 20.$$

$$L \tan. \tfrac{1}{2}(C-B) = L \sin. \tfrac{1}{2}(c-b) + L \operatorname{cosec}. \tfrac{1}{2}(c+b) + L \cot. \frac{A}{2} - 20.$$

Thus, in the triangle given,

c	119° 42′ 15″
b	108 4 15
$c+b$	227 46 30
$\frac{1}{2}(c+b)$	113 53 15
$c-b$	11 38 0
$\frac{1}{2}(c-b)$	5 49 0

L cos. $\frac{1}{2}(c-b)$.	9·997758	L sin. $\frac{1}{2}(c-b)$.	9·005805
L sec. $\frac{1}{2}(c+b)$.	10·392607	L cosec. $\frac{1}{2}(c+b)$	10·038891
L cot. $\frac{A}{2}$	10·110905	L cot. $\frac{A}{2}$ . . .	10·110905
(sum)	30·501270	(sum)	29·155601
	20·000000		20·000000
(difference) L tan. $\frac{1}{2}(C+B)$	10·501270	(difference) L tan. $\frac{1}{2}(C-B)$	9·155601

therefore $\frac{1}{2}(C+B) = 107° 30' 0''$. $\frac{1}{2}(C-B) = 8° 8' 30''$.

$$\begin{aligned}\tfrac{1}{2}(C + B) &= 107° \; 30' \; 0'' \\ \tfrac{1}{2}(C - B) &= 8 \;\; 8 \;\; 30 \\ C &= 115 \;\; 38 \;\; 30 \\ B &= 99 \;\; 21 \;\; 30\end{aligned}$$

It will be noticed that 107° 30′, and not 72° 30′, is taken as the value of $\frac{1}{2}(C+B)$. For we know that $\frac{1}{2}(C + B)$ must be greater than 90°, because, as explained in Part II. art. 48, $\frac{1}{2}(C + B)$ and $\frac{1}{2}(c + b)$ must be *of the same affection*, that is, must be both less or both greater than 90°; we therefore select the greater of the two values.

Examples.—XVIII.

1. In the spherical triangle ABC find the third side, having given

(1) $A = 96° 32' 0''$, $b = 76° 42' 0''$, $c = 89° 10' 30''$.
(2) $A = 50° 0' 0''$, $b = 70° 45' 15''$, $c = 62° 10' 15''$.
(3) $a = 100° 8' 45''$, $b = 98° 10' 0''$, $C = 88° 24' 30''$.
(4) $b = 118° 2' 15''$, $c = 120° 18' 30''$, $A = 27° 22' 30''$.
(5) $a = 87° 10' 15''$, $b = 62° 36' 45''$, $C = 102° 58' 30''$.
(6) $a = 69° 19' 15''$, $b = 78° 59' 15''$, $C = 110° 48' 45''$.

2. In the spherical triangle ABC find directly by Napier's Analogies the remaining angles of the triangle, having given

(1) $A = 85° 31' 15''$, $b = 49° 36' 0''$, $c = 100° 17' 30''$.
(2) $a = 109° 15' 30''$, $b = 93° 26' 30''$, $C = 53° 21' 30''$.

Case III.

20. *Having given two sides of a spherical triangle, and the angle opposite to one of them, to find the other parts.* (See Part II., arts. 52, 53.)

In the spherical triangle ABC let $a = 80° 5' 0''$, $b = 70° 10' 30''$, and $A = 33° 15'$; required the other parts.

(1) *To find the angle* B.

Since
$$\sin. B = \frac{\sin. b}{\sin. a} \sin. A,$$

therefore L sin. B = L sin. A + L sin. b + L cosec. a − 20.

L sin. 33° 15′ 0″ . . .	9·739013
L sin. 70° 10′ 30″ . . .	9·973466
L cosec. 80° 5′ 0″ . . .	10·006538
(sum)	29·719017
	20·000000
(difference) L sin. B . .	9·719017

There are two angles less than 180°, which correspond to the given logarithm, viz. 31° 34′ 30″ and 148° 25′ 30″, and we have to determine which of the two is the angle sought, or whether both values are admissible.

Since b is less than a, the angle B must be less than A (Part II., art. 35), that is, less than 33° 15′.

The value 148° 25′ 30″ is therefore plainly inadmissible, and 31° 34′ 30″ is the only value of B.

(2) *To find the side* c.

We have now the two sides a, b, and the angles opposite to these sides, A, B. Hence, by Napier's Analogies,

$$\tan. \frac{c}{2} = \frac{\cos. \frac{1}{2}(A + B)}{\cos. \frac{1}{2}(A - B)} \tan. \tfrac{1}{2}(a + b);$$

therefore L tan. $\frac{c}{2}$ = L cos. $\frac{1}{2}$(A + B) + L sec. $\frac{1}{2}$(A − B) + L tan. $\frac{1}{2}(a + b)$ − 20.

A	33° 15 0″	a . .	80° 5′ 0″
B	31 34 30	b . .	70 10 30
A + B . .	64 49 30	$a+b$.	150 15 30
$\frac{1}{2}$(A + B) .	32 24 45	$\frac{1}{2}(a+b)$	75 7 45
A − B . .	1 40 30		
$\frac{1}{2}$(A − B) .	0 50 15		

L cos. $\frac{1}{2}$ (A + B) . . .	9·926451
L sec. $\frac{1}{2}$ (A − B) . . .	10·000046
L tan. $\frac{1}{2}$ $(a+b)$ . . .	10·575879
(sum)	30·502376
	20·000000
(difference) L tan. $\frac{c}{2}$ . .	10·502376

Therefore $\frac{c}{2} = 72° 32' 30''$,

and $c = 145° 5' 0''$.

(3) *To find the angle C.*

Having now the three sides a, b, c, we may find C as in Case I., art. 18.

Thus L hav. C = log. cosec. a + log. cosec. b + $\frac{1}{2}$ L hav. $(c + \overline{a - b})$ + $\frac{1}{2}$ L hav. $(c - \overline{a - b})$.

a	80° 5′ 0″	log. cosec. a. . .	·006538
b	70 10 30	log. cosec. b	·026534
$a-b$ . .	9 54 30	$\frac{1}{2}$ L hav. $(c+\overline{a-b})$.	4·989581
c	145 5 0	$\frac{1}{2}$ L hav. $(c-\overline{a-b})$.	4·965902
$c+\overline{a-b}$.	154° 59′ 30″	(sum) L hav. C . .	9·988555
$c-\overline{a-b}$.	135° 10′ 30″		

therefore C = 161° 26′ 0″.

21. In the preceding example, as we have seen, one triangle only was possible for the given data. That which follows supplies an illustration of the ambiguities discussed in articles 52 and 53 of Part. II.

In the spherical triangle ABC having given $a = 50° 45' 15''$, $b = 69° 12' 45''$, $A = 44° 22' 15''$, find the angle B.

As before,

L sin. B = L sin. A + L sin. b + L cosec. a − 20.

L sin. 44° 22′ 15″ . . .	9·844633
L sin. 69 12 45 . . .	9·970767
L cosec. 50 45 15 . . .	10·111013
(sum)	29·926413
	20·000000
(difference) L sin. B . .	9·926413

The two angles less than 180° which correspond to the tabular logarithm 9·926413 are 57° 34′ 45″ and 122° 25′ 15″. From the data we see that the angle B must be greater than the angle A, a condition which is satisfied by either of the values obtained, and two triangles are therefore possible.

In the example given b is 69° 12′ 45″, and 180° − b is therefore 110° 47′ 15.″ Since a, the side opposite to the given angle, is in this case 50° 45′ 15″, and therefore does not lie between those values, it might have been anticipated that two solutions would be obtained. (See Part II., art. 53.)

Having now in each of the two triangles two sides, and the angles respectively opposite to those sides, the remaining parts in each triangle may be determined by the processes of the preceding article.

EXAMPLES.—XIX.

1. In the spherical triangle ABC find the angle B, having given

(1) a = 119° 21′ 0″, b = 50° 26′ 0″, A = 108° 35′ 30″.

(2) a = 58° 27′ 30″, b = 117° 46′ 30″, A = 35° 17′ 15″.

2. In the spherical triangle ABC find the other parts, having given

a = 87° 36′ 0″, b = 45° 31′ 30″, A = 75° 27′ 30″.

CASE IV.

22. *Having given two angles of a spherical triangle, and a side opposite to one of them, to find the other parts.* (See Part II., art. 54.)

In the spherical triangle ABC let A = 75° 31′ 0″, B = 110° 16′ 45″, a = 83° 13′ 30″.

To find the side b.

Since $$\sin. b = \frac{\sin. B}{\sin. A} \sin. a,$$

therefore $L \sin. b = L \sin. B + L \sin. a + L \text{cosec.} A - 20$

L sin. 110° 16′ 45″	. . .	9·972210
L sin. 83 13 30	. . .	9·996957
L cosec. 75 31 0	. . .	10·014026
(sum)		29·983193
		20·000000
(difference) L sin. b	. .	9·983193

Therefore $b = 74° 9' 45''$, or $105° 50' 15''$. And since b must in this case be greater than a, which is $83° 13' 30''$, the second of these values is clearly the only one admissible; that is, $b = 105° 50' 15''$.

By a test similar to that applied in Case III. we may ascertain whether the given conditions may be expected to furnish one result or two by considering whether the angle A, opposite to the given side, falls between B and $180° - B$. In the present instance these angles are $110° 16' 45''$ and $69° 43' 15''$. The value of A, $75° 31' 0''$, lies between these limits, and consequently only one value of a was to be looked for.

We have now in the triangle ABC two sides and the angles opposite to those sides; the remaining parts of the triangle may therefore be determined by the methods explained under Case III.

EXAMPLES.—XX.

In the spherical triangle ABC find the side b, having given

(1) $a = 81° 37' 30''$, $A = 78° 29' 0''$, $B = 75° 49' 30''$.
(2) $a = 105° 30' 0''$, $A = 99° 36' 30''$, $B = 86° 25' 15''$.

CASE V.

23. *Having given two angles of a spherical triangle and the included side, to find the other parts.* (See Part II., art. 55.)

In the spherical triangle ABC let $A = 113° 33' 30''$, $B = 51° 30' 30''$, $c = 60° 18' 0''$; it is required to find the other parts.

To find the angle C.

Let A′B′C′ be the polar triangle to ABC.

Then A + a' = 180°, B + b' = 180°, c + C′ = 180° (Part II., art. 26).

A .	113° 33′ 30″	B .	51° 30′ 30″	c .	60° 18′ 0″
	180°		180°		180°
(difference) a' .	66° 26′ 30″	b' .	128° 29′ 30″	C′ .	119° 42′ 0″

Hence in the triangle A′B′C′, having given two sides and the included angle, we may find the third side, as in Case II.

L sin. a' . . .	9·962205	b' . . .	128° 29′ 30″
L sin. b' . . .	9·893595	a' . . .	66 26 30
L hav. C′ . . .	9·873744	$b'-a'$.	62 3 0
	29·729544		
	20·000000		
(difference) L hav. θ . .	9·729544		

therefore θ = 94° 11′ 0″

tab. vers. θ . . .	1072948
tab. vers. $(b' - a')$ . . .	531299
tab. vers. c' . . .	1604247

therefore c' . . . 127° 10′ 29″

180 0 0

(difference) C . . . 52° 49′ 31″

To obtain the remaining two sides, a, b, of the triangle ABC, we first determine the angles A′, B′ of the polar triangle, as in Case I. Thus:

b' . . .	128 29 30	log. cosec. c' .	·098654
c' . . .	127° 10′ 30″	log. cosec. b' .	·106405
$b'-c'$ . . .	1 19 0	$\frac{1}{2}$L hav. $(a' + \overline{b'-c'})$	4·746248
a' . . .	66 26 30	$\frac{1}{2}$L hav. $(a' - \overline{b'-c'})$	4·731009
$a' + \overline{b'-c'}$ . . .	67 45 30	L hav. A′ . . .	9·682316
$a' - \overline{b'-c'}$ . . .	65 7 30	∴ A′ . .	87° 50′ 45″
			180
		(difference) a =	92° 9′ 15″

And in the same way the remaining side b may be determined.

CASE VI.

24. *Having given the three angles of a spherical triangle, to find the three sides.* (See Part II., art. 56.)

In the spherical triangle ABC let A = 101° 17′ 30″, B = 109° 52′ 0″, C = 102° 31′ 0″; it is required to find the sides.

Then if A′B′C′ be the polar triangle, we shall have A + a' = 180°, B + b' = 180°, C + c' = 180°.

A . .	101° 17′ 30″	B . .	109° 52′ 0″	C . .	102° 31′ 0″
	180		180		180
a' . .	78° 42′ 30″	b' . .	70° 8′ 0″	c' . .	77° 29′ 0″

To find the side a.

c' . . .	77° 29′ 0″	log. cosec. c' . . .	·010447
b' . . .	70 8 0	log. cosec. b' . . .	·026648
$c' - b'$ . . .	7 21 0	$\frac{1}{2}$L hav. $(a' + \overline{c' - b'})$.	4·834054
a' . . .	78 42 30	$\frac{1}{2}$L hav. $(a' - \overline{c' - b'})$.	4·765896
$a' + \overline{c' - b'}$ . . .	86° 3′ 30″	(sum) L hav. A′ . . .	9·637045
$a' - \overline{c' - b'}$ . . .	71° 21′ 30″	∴ A′ =	82° 21′ 45″
			180
		(difference) a =	97° 38′ 15″

And in the same manner the other sides of the triangle may be determined.

EXAMPLES.—XXI.

In the spherical triangle ABC find the other parts, having given

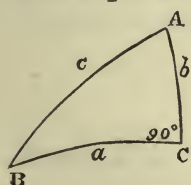
(1) A = 81° 24′ 45″, B = 61° 31′ 45″, C = 102° 59′.
(2) A = 78° 36′ 30″, B = 83° 16′ 15″, C = 98° 34′ 30″.
(3) A = 69° 37′ 45″, B = 88° 35′ 45″, C = 121° 37′ 30″.

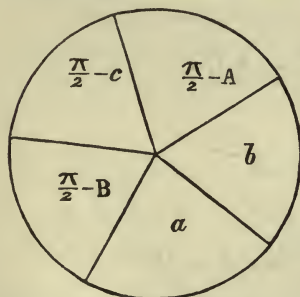
CHAPTER VI.

THE SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

(See Part II., Chapter V.)

25. As has been explained in Part II., arts. 57—63, when in a right-angled spherical triangle any two parts beside the right angle are given, each of the remaining parts may be obtained by the addition or subtraction of two logarithms.

The requisite formulæ are furnished by the system of rules called *Napier's Rules of Circular Parts.*

Let ABC be a spherical triangle, having the angle C a right angle.

The two sides a, b and the complements of the side c and of the angles A, B are known as the circular parts, and are usually arranged round a circle, as described in Part II., art. 60. Since there are five parts in all, it must happen that the three involved in a particular formula either all lie together, or else, two being together, that the third is separated from each of them.

When all three lie together the part which is between the other two is the *middle part,* the parts on either side being called the *adjacent parts.* When one of the three is separated from the other two it is still called the *middle part,* and the other two parts are called the *opposite parts.*

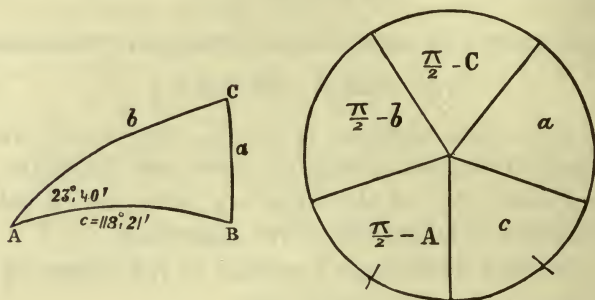
All the formulæ necessary for the complete solution of the triangle are now furnished by the rules.

Sine *middle* part = product of tangents of *adjacent* parts.

Sine *middle* part = product of cosines of *opposite* parts.

The student should be careful to write over each function of

a known part its appropriate sign, in order that he may determine the sign belonging to the ratio from which the unknown part is to be obtained.

Example.—In the spherical triangle ABC, having given $c = 118° 21'$, $A = 23° 40'$, $B = 90°$, find the remaining parts.

It is convenient to mark the two parts given, as shown in the figure.

(1) *To find the side a.*

The three parts involved are $a, c, \frac{\pi}{2} - A$, of which c is the middle part, a and $\frac{\pi}{2} - A$ being the adjacent parts.

Hence

$$\sin. c = \tan. a \tan. \left(\frac{\pi}{2} - A\right),$$

or,

$$\sin. c = \tan. a \cot. A.$$

Therefore

$$\overset{+}{\tan.} a = \overset{+}{\sin.} c \ \overset{+}{\tan.} A,$$

and

$$L \tan. a = L \sin. c + L \tan. A - 10.$$

And the signs of sin. c and tan. A being in each case +, the sign of tan. a also is +.

(2) *To find the side b.*

The three parts in this case are $\frac{\pi}{2} - b, \frac{\pi}{2} - A, c$, of which $\frac{\pi}{2} - A$ is the middle part, and $c, \frac{\pi}{2} - b$ are the adjacent parts.

Hence $$\sin.\left(\frac{\pi}{2}-\text{A}\right)=\tan. c \tan.\left(\frac{\pi}{2}-b\right),$$

or, $$\cos. \text{A} = \tan. c \cot. b.$$

Therefore $$\overset{-}{\cot. b} = \overset{+}{\cos. \text{A}}\ \overset{-}{\cot. c},$$

and $$\text{L} \cot. b = \text{L} \cos. \text{A} + \text{L} \cot. c - 10.$$

In this case cos. A is +, and cot. c is —. Therefore cot. b also is —, which shows that b is greater than 90°.

(3) *To find the angle* C.

Here $\left(\frac{\pi}{2}-\text{C}\right)$ is the middle part, c, $\frac{\pi}{2}-\text{A}$ being the opposite parts. Therefore

$$\sin.\left(\frac{\pi}{2}-\text{C}\right)=\cos. c \cos.\left(\frac{\pi}{2}-\text{A}\right),$$

or, $$\overset{-}{\cos. \text{C}} = \overset{-}{\cos. c}\ \overset{+}{\sin. \text{A}},$$

and $$\text{L} \cos. \text{C} = \text{L} \cos. c + \text{L} \sin. \text{A} - 10.$$

Here again cos. c is —, so that cos. C also is —, and C must be greater than 90°.

The necessary logarithms may now be set down in order; then, adding each pair and rejecting 10 from the characteristic in each sum, we shall obtain the logarithms of the parts required.

(1)		(2)	
L sin. c . .	9·944514	L cot. c . .	9·732043
L tan. A .	9·641747	L cos. A . .	9·961846
(sum) L tan. a . .	9·586261	(sum) L cot. b . .	9·693889

(3)

L cos. c . . .	9·676562
L sin. A . .	9·603594
(sum) L cos. C . .	9·280156

therefore $a=21°\ 5'\ 30''$, $b=180°-63°\ 42'\ 15''=116°\ 17'\ 45''$,
$\text{C}=180°-79°\ 0'\ 45''=100°\ 59'\ 15''$.

EXAMPLES.—XXII.

In the right-angled triangle ABC find the other parts, having given

(1) $b = 60° 10' 0''$, $c = 100° 0' 0''$, $A = 90°$.
(2) $B = 100° 0' 0''$, $C = 87° 10' 0''$, $A = 90°$.
(3) $c = 46° 18' 30''$, $B = 34° 27' 30''$, $A = 90°$.
(4) $a = 85° 17' 0''$, $b = 102° 26' 15''$, $A = 90°$.
(5) $a = 100° 42' 0''$, $B = 78° 10' 0''$, $A = 90°$.
(6) $c = 53° 14' 15''$, $A = 91° 26' 0''$, $B = 90°$.
(7) $a = 120° 18' 45''$, $b = 101° 9' 0''$, $C = 90°$.
(8) $B = 72° 19' 0''$, $b = 50° 50' 0''$, $A = 90°$.

Note.—Example (8) affords an illustration of the ambiguity discussed in Part II., art. 63. A diagram may easily be constructed similar to the one given in that place, from which it will at once be seen how it is that a double set of results may be found to satisfy the data of the question.

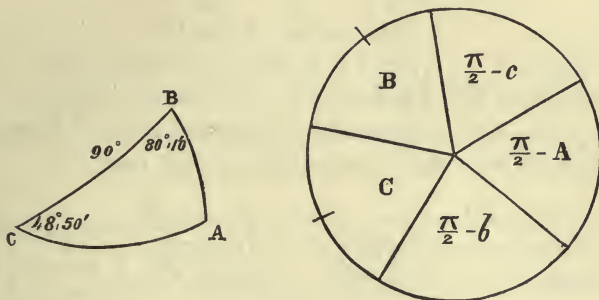
CHAPTER VII.

THE SOLUTION OF QUADRANTAL SPHERICAL TRIANGLES.

(See Part II., Chapter VI.)

26. If one side of the spherical triangle ABC, as c, be a quadrant, the rules of Circular Parts, given in art. 25, may be used in its solution, the five circular parts in this instance being A, B, and the complements of a, b, and C.

It must be remembered, as pointed out in Part II., art. 66, that whenever the two adjacent parts, or the two opposite parts, are both sides or both angles, the sign $-$ must be attached to their product.

Example.—In the spherical triangle ABC let $a = 90°$, $B = 80° 10'$, and $C = 48° 50$; it is required to find the other parts.

(1) *To find the angle* A.

Here $\left(\frac{\pi}{2} - A\right)$ is the middle part; B, C are the opposite parts.

Thus we have

$$\sin.\left(\frac{\pi}{2} - A\right) = -\cos. B \cos. C,$$

or,

$$\overset{-}{\cos. A} = -\overset{+}{\cos. B}\ \overset{+}{\cos. C};$$

therefore $\quad L \cos. A = L \cos. B + L \cos. C - 10.$

And the sign of cos. B and of cos. C being in each case $+$, it follows that the sign of cos. A must be $-$, and A therefore greater than 90°.

(2) *To find the side b.*

In this case C is the middle part, B and $\left(\frac{\pi}{2} - b\right)$ the adjacent parts.

Therefore

$$\sin. C = \tan.\left(\frac{\pi}{2} - b\right) \tan. B,$$

or, $\quad \overset{+}{\cot. b} = \overset{+}{\sin. C}\ \overset{+}{\cot. B};$

and $\quad L \cot. b = L \sin. C + L \cot. B - 10;$

and the sign of cot. b being $+$, b is less than 90°.

(3) *To find the side c.*

Here B is the middle part, C and $\left(\frac{\pi}{2} - c\right)$ the adjacent parts.

Hence

$$\text{sin. B} = \text{tan. C tan.} \left(\frac{\pi}{2} - c\right),$$

$$\begin{array}{lc} & \quad + \qquad + \qquad + \\ \text{or,} & \text{cot. } c = \text{sin. B cot. C}; \\ \text{therefore} & \text{L cot. } c = \text{L sin. B} + \text{L cot. C} - 10, \end{array}$$

and the sign of cot. c is $+$.

We have now only to add the several pairs of logarithms, and reject 10 from the characteristic in each case.

(1)		(2)	
L cos. B . .	9·232444	L cot. B . .	9·238872
L cos. C . .	9·818392	L sin. C . .	9·876678
L cos. A . .	9·050836	L cot. b . .	9·115550

(3)	
L sin. B . .	9·993572
L cot. C . .	9·941713
L cot. c . .	9·935285

therefore $A = 180° - 83° 32' 45'' = 96° 27' 15''$. $b = 82\ 34$.

$c = 49° 15' 15''$.

EXAMPLES.—XXIII.

In the spherical triangle ABC find the other parts, having given

(1) $A = 100°$, $c = 50° 10'$, $a = 90$.
(2) $B = 45°$, $c = 72°$, $a = 90°$.
(3) $B = 80° 10'$, $C = 50° 2'$, $a = 90°$.
(4) $A = 72° 49' 45''$, $b = 47° 44' 30''$ $a = 90°$.
(5) $c = 49° 23' 45''$, $b = 76° 41'$, $a = 90°$.
(6) $a = 60° 10' 15''$, $b = 80° 20' 30''$, $c = 90°$.

Note.— In the solution of right-angled and quadrantal spherical triangles it will be well for the beginner to arrange the several parts of the triangle in the five compartments of a circle, as has been done in this and in the previous chapter. As, however, he acquires familiarity with the Rules, he will probably find that he is able to dispense with the circle, and deduce the appropriate relation between the three parts involved directly from inspection of the triangle.

MISCELLANEOUS EXAMPLES IN PLANE TRIGONOMETRY.

1. An observer on the bank of a river finds that the elevation of the top of a tower on the opposite bank, known to be 216 feet high, is 47° 56′, the height of his eye from the ground being 5 feet. Find the distance of the observer from the foot of the tower.

2. A straight line, AD, 100 feet in height, stands at right angles to another straight line BDC at the point D. At B the angle subtended by AD is 36° 48′, at C the angle is 54° 30′. Find the length of BDC.

3. A field is in the form of a right-angled triangle. The base is 200 feet, and the angle adjacent to the base 67°. How long will a man take to walk round it at the rate of four miles an hour?

4. Two points, B and C, are 100 feet apart, and a third point, A, is equally distant from B and C. What must be the distance of A from each of the other two points in order that the angle BAC may be 150°?

5. A May-pole being broken off by the wind, its top struck the ground at a distance of 15 feet from the foot of the pole. Find the original height of the pole, if the length of the broken portion is 39 feet.

6. A ladder 36 feet long reaches a window 30·7 feet from the ground on one side of a street, and when turned over about its foot just reaches another window 18·9 feet high on the other side of the street. Find the breadth of the street.

7. From the bottom of a tower a distance, AB, is measured in the horizontal plane, and found to be 50 yards, and at A the angle BAC is observed to be 25° 17′. Required the height of the tower BC.

8. To determine the distance of a ship at anchor at C a distance, AB, of 1,000 yards was measured on the shore, and the angles CAB, CBA were found to be 32° 10′ and 83° 18′ respectively. Find the distance of the ship from A.

9. Two ships, A, B, are anchored two miles apart. At A the angle between the other ship and an object, C, on shore is found to be 85° 10′, at B the angle between C and the other ship is 82° 45′. Find the distances of the two ships from C.

10. The two sides, AB, BC, of the right-angled triangle ABC are 18 and 24. Find the length of the perpendicular let fall from the right angle upon the hypothenuse.

11. To determine the height, AB, of a tower inaccessible at the base, two stations, C, D, are chosen in a horizontal plane, so that A, C, and D are in the same vertical plane. The distance CD is 100 yards, the angle ACB is 46° 15′, and BDA 31° 20′. Find the height AB.

12. From the decks of two ships at C and D, 880 yards apart, the angle of elevation of a cloud at A, in the same vertical plane as C, D, is observed, and at C found to be 35°, at D 64°. Find the height of the cloud above the surface of the sea, the height of eye in each case being 21 feet.

13. A tower subtended 39° to an observer stationed 200 feet from the base. Find its height, and also the angle which it will subtend to an observer at 350 feet from its base.

14. To determine the distance between two ships at sea an observer noted the interval between the flash and report of a gun fired on board each ship, and measured the angle which the two ships subtended. The intervals were 4 seconds and 6 seconds respectively, and the angle 48° 42′. Find the distance of the ships from each other, having given the velocity of sound 1,142 feet per second.

15. From the top of a ship's mast, 80 feet above the water, the angle of depression of another ship's hull was observed, and found to be 20°. Required the distance between the two ships.

16. Two monuments are 50 feet and 100 feet in height respectively, and the line joining their tops makes with the horizontal plane an angle of 37°. Find their distance apart.

17. To determine the distance between two ships at anchor,

C, D, a base AB is measured on the beach, and found to be 670 yards. The following angles were then observed at the extremities of the base: at A the angle BAD 40° 16′, BAC 97° 56′, at B, ABC 42° 22′, and ABD 113° 29′. Find the distance of the two ships apart.

18. To determine the distance of two forts, C, D, at the mouth of a harbour, a boat is placed at A, with its bow towards a distant object E, and the angles CAD, DAE, are observed and found to be 22° 17′ and 48° 1′ respectively. The boat is then rowed to B, a distance of 1,000 yards, directly towards E, and the angles CBD, DBE are observed to be 53° 15′ and 75° 43′ respectively. Find the distance CD.

19. To determine the height of an object, EB, on the top of an inaccessible hill, the angle of elevation, ACE, of the top of the hill was observed, and found to be 40°, and that of the top of the object, ACB, was found to be 51°. After walking a distance of 100 yards in a horizontal line directly away from the object, the observer found the angle of elevation of the top of the object, ADB, to be 33° 45′. Find the height of the object.

20. To determine the distance from an inaccessible object at O, *without observing any angles*, a straight line, AB, of 500 yards, was measured, so that O was visible from each extremity. From A, B, in a direct line from O, AC and BD were measured, each equal to 100 yards. Finally the distances AD and BC were measured. The former was 550 yards, the latter 560 yards. Find the distances of the object from A and B.

21. The angle of elevation of a tower 100 feet high due north of an observer was 50°. What will be its elevation when the observer has walked due east 300 feet?

22. The elevation of a balloon was observed at a certain station to be 20°, and its bearing was N.E. At a second station 4,000 yards due south of the former one its bearing was N. by E. Find its height.

23. From a window which seemed to be on a level with the bottom of a steeple the angle of elevation of the top of the steeple was 40°. At another window, 18 feet vertically above the former, the angle of elevation was 37° 30′. Find the height of the steeple.

24. At B, the top of a castle which stood on a hill near the seashore, the angle of depression, HBS, of a ship at anchor was 4° 52′, and at R, the bottom of the castle, its depression, NRS, was 4° 2′. Find the height of the top of the building above the level of the sea, the height of the castle itself being 54 feet.

25. In order to find the breadth of a river a base line of 500 yards was measured in a straight line close to one side of it, and at each extremity of the base the angle subtended by the other end and a tree upon the opposite bank were measured. These angles were 53° and 79° 12′ respectively; find the breadth of the river.

26. The elevation of the top of a spire at one station, A, was 23° 50′ 15″, and the horizontal angle at this station between the spire and another station, B, was 93° 4′ 15″. The horizontal angle at B was 54° 28′ 30″, and the distance between the stations 416 feet; what was the height of the spire?

27. In order to find the distance of a battery at B from a fort at F, distances BA, AC were measured to points A, C, from which both the fort and battery were visible, the former distance being 2,000, and the latter 3,000 yards. The following angles were then observed: BAF = 34° 10′, FAC = 54° 42′, and FCA = 80° 10′. From these data find the distance of the fort from the battery.

28. From a ship sailing along a coast a headland, C, was observed to bear N.E. by N. After the ship had sailed E. by N. 15 miles the headland bore W.N.W. Find the distance of the headland at each observation.

29. A cape, C, bore from a ship N.W., and a headland, H, bore N.N.E. $\frac{1}{2}$ E. After the ship had sailed E. by N. $\frac{1}{2}$ N. 23 miles the cape bore W.N.W. and the headland N. by W. $\frac{1}{2}$ W. Find the bearing and distance of the cape from the headland.

30. From a ship sailing N.W. two islands appeared in sight, one bearing W.N.W., the other N. When the ship had sailed six miles farther they bore W. by S. and N.E. respectively. Find their bearing and distance from each other.

31. A ship was 2,640 yards due south of a lighthouse. After the ship had sailed N.W. by N. 800 yards the angle of elevation of the top of the lighthouse was 5° 25′. Find its height.

32. A church, C, bears from a battery, B E.N.E., 960 yards distant. How must the church bear from a ship at sea which runs in until the battery is due north, 2,000 yards distant?

33. What angle will a tower subtend at a distance equal to six times the height of the tower? Where must the observer station himself that the angle of elevation may be double the former angle?

34. A May-pole was broken by the wind, and its top struck the ground 20 feet from the base. Being again fixed it was broken a second time 5 feet lower down, and its top reached the ground at a point 10 feet farther than before. Find the height of the pole.

35. The summit, A, of a hill bore east from a spectator at B, and E.N.E. from a spectator at C, a point due south of B. The angle of elevation of the point A at B being 20°, find its elevation at C.

36. An observer finds the angle of elevation of a tower at a point B to be 23° 18′. After walking from B 300 feet, in a direction at right angles to the line joining B with the foot of the tower, he found the elevation to be 21° 16′. Required the distance of the tower from B.

37. The distance between two objects, C and D, is known to be 6,594 yards. On one side of the line CD there are two stations, A, B, at which angles are observed. The angle CAD is 85° 46′, DAB 23° 56′, CBD 68° 2′, and CBA 31° 48′. From these observations find the distance between A and B.

38. The area of a triangle is 6 square feet, and two of its sides are 3 feet and 5 feet. Find the third side.

39. Find the area of a regular octagon, the side of which is 16 yards.

40. The area of a regular decagon is 3233·5 square yards. Find a side.

41. If at the top of a mountain the true depression of the horizon is 1° 31′, find the height of the mountain, supposing the earth to be a sphere of diameter 8,000 miles.

42. A flagstaff, 12 feet high, on the top of a tower, subtended an angle of 48′ 20″ to an observer at a distance of 100 yards from the foot of the tower. Find the height of the tower.

43. Walking along a road I observed the elevation of a tower to be 20°, and the angular distance of its top from an object in the road was 30°. The shortest distance from the tower to the road being 200 feet, find the height of the former.

44. The sides of a triangle were 6, and the angles were to each other :: 1 : 2 : 3. Find the sides.

45. The perimeter of a triangle is 100 yards, and the angles are to each other in the proportions of 1, 2, 4. Find the sides of the triangle.

46. The perimeter of a right-angled triangle is 24 yards, and one of its angles is 30°. Find the sides.

47. In the plane triangle ABC the side a is 400 feet, and the sum of b and c is 600 feet, the angle A being 80°. Find b and c.

48. The perimeter of a right-angled triangle is 24 feet, and its base is 8 feet. Find the other sides.

49. At the distance of 80 feet from a steeple the angle made by a line drawn from its top to the place of observation was double that made by a line drawn from the top to a point 250 feet from the steeple on the same level. Find the height of the steeple.

50. In the plane triangle ABC, the side c is 70 feet, $a - b = 13$ feet, and $A - B = 20°$. Solve the triangle.

MISCELLANEOUS EXAMPLES IN SPHERICAL TRIGONOMETRY.

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1. Having given the Sun's meridian altitude 70° (zenith north of the Sun) and its declination 20° N., required the latitude of the place.

2. Having given the Sun's meridian altitude 70° (zenith north) and its declination 5° S., required the latitude.

3. Having given a star's meridian altitude 70° (zenith south) and declination 25° N., required the latitude.

4. Having given the Sun's meridian altitude 30° (zenith south) and declination 10° N., required the latitude.

5. Find the maximum altitude attained by a body of declination 20° N. in latitude 40° N.

6. Having given the Sun's meridian altitude 30° (zenith south) at a place in latitude 50° S., find its declination.

7. The meridian altitude of a star at a place on the Equator is 57°; find its declination (zenith north of the star).

8. The meridian altitudes of a circumpolar star at its superior and inferior transits were 70° and 20° respectively; required the latitude.

9. A circumpolar star passes the zenith of a place, and its altitude at the inferior transit is 20°; required the latitude.

10. If the altitude of a circumpolar star at its inferior transit is equal to its zenith distance at its superior transit, required the latitude.

11. In latitude 60° N. find the altitudes of a star at the inferior and superior transits, the declination being 40° N.

12. What is the declination of a star that passes the zenith of a place in latitude 50° 48′ N., and what will be its altitude at the inferior transit?

13. Find the latitude of a place at which the Sun's centre just touches the horizon without setting on the longest day.

14. In latitude 50° 48′ N. the altitude of the Sun was 46° 20′ (west of the meridian) and its declination was 23° 27′ 45″ N.; find the azimuth and the apparent time.

15. The azimuth of a heavenly body was N. 111° 51′ W., its altitude at the same time was 46° 20′, and declination 23° 27′ 45″ N.; find the apparent time.

16. Find the altitude of a star, whose hour-angle is 2h. 32m. and declination 16° N., at a place in latitude 50° 48′ N.

17. In latitude 50° 48′ N., the Sun's declination being 12° 29′ N., find his azimuth at 2h. 53m. 1s. A.M., apparent time.

18. Having given the Sun's altitude 42° 30′, declination 22° 10′ N., and azimuth S. 57° 45′ W., find the latitude.

19. Having given the Sun's altitude 37° 20′, hour-angle 2h. 15m., and declination 22° 30′ N., find the latitude (zenith north of the Sun).

20. Having given the Sun's altitude 30°, when due west, and its declination 20° N., find the latitude.

21. Having given the Sun's declination 23° 27′ 45″, and the latitude of the place 50° 48′ N., find the time when he will be on the prime vertical, and the altitude at the time.

22. Two stars are due east at the same time at a place in latitude 50° 48′ N.; their altitudes are 20° and 40°; find the difference of their hour-angles.

23. Having given the Sun's altitude at six o'clock 18° 45′, and declination 20° 4′ N., find the latitude.

24. Having given the latitude of the place 50° 48′ N., and the Sun's declination 23° 27′ 45″ N., find the altitude and azimuth at six o'clock.

25. Find the apparent time of sun-rise at a place in latitude 50° 48′ N. when the amplitude is E. 10° S., neglecting the effects of refraction.

26. Having given the Sun's amplitude W. 37° 30′ N., and declination 15° 12′ N., required the latitude.

27. Having given the latitude of the place 50° 48′ N., and the Sun's declination 18° 28′ N.; find the amplitude and the length of the day.

28. Where will the Sun rise in latitude 50° 48′ N. when the day is 14 hours long?

29. Having given the Sun's altitude 22° 56′, the hour-angle 3h., and the declination 0°, find the latitude.

30. The right ascension of the Sun is 4h. 10m. 20s., and the obliquity of the ecliptic 23° 27′ 45″; find the declination and longitude.

31. The right ascension of a heavenly body is 2h. 59m. 37s., the declination 21° 27′ 45″ N.; find its latitude and longitude.

32. The latitude of a heavenly body was 46° 6′ 15″ N., and the longitude 234° 36′ 30″; find the declination and right ascension.

33. Two places have the same latitude, 45° N., and their difference of longitude is 10° 36′; find their distance apart, measured on the parallel of latitude passing through them.

34. Find the distance between Portsmouth and Buenos Ayres, measured upon the arc of the great circle passing through these places, having given

Lat. Portsmouth. . . .	50°	48′	N.
,, Buenos Ayres . . .	34	37	S.
Long. Portsmouth . . .	1	6	W.
,, Buenos Ayres . .	58	24	W.

35. A ship from latitude 50° 10′ N. starts on a great circle, sailing S. 45° W. What course will she be steering after following the arc of the great circle for 100 miles?

36. What is the highest latitude attained by a ship sailing on the arc of a great circle from Port Jackson to Cape Horn, their latitudes being 33° 51′ S. and 55° 58′ S. respectively, and the difference of longitude 140° 27′?

37. Required the Sun's depression below the horizon at 7h. P.M., when the declination is 10° 15′ S., and the latitude of the place 50° 48′ N.

38. Determine the azimuth of the two stars Aldebaran and Pollux when on the same vertical circle, the latitude of the place being 25° N.; the R.A. and declination of the former star being 4h. 26m. 46s. and 16° 11′ N.; of the latter 7h. 35m. 13s. and 28° 24′ 30″ N.

39. In a certain latitude (zenith N.) the Moon's true altitude was 18° 2′ 30″ (east of meridian) when upon the same vertical circle as a star whose R.A. and declination were 9h. 59m. 42s. and 12° 45′ 45″ N. The Moon's R.A. and declination being 12h. 35m. 54s. and 1° 42′ 30″ S., required the latitude.

The problems hitherto given may be regarded as exercises in the practical solution of spherical triangles. Those which follow require for the most part a greater degree of mathematical skill, and are therefore distinguished by an asterisk:—

* **40**. The altitude of a star when due east was 20°, and it rose E. by N. Find the latitude.

* **41**. The altitude of a star when due east was 10°, and when due south 40°; find the latitude.

* **42**. Having given the altitude of the Sun when due west, and at six o'clock, to find the latitude and declination.

Example.—Altitude when west 27° 24′, at six o'clock 14° 43′ 30″.

* **43**. Having given the Sun's altitude at six o'clock, and his amplitude, to find the latitude of the place and declination of the sun.

Example.—Altitude at six o'clock 14° 43′ 30″, amplitude W. 30° 44′ 30″ N.

* **44**. Having given the Sun's altitude at six o'clock, and the hour-angle at setting, to find the latitude and declination.

Example.—Altitude at six o'clock 14° 43′ 30″; hour-angle at setting 7h. 35m. 22s.

* **45**. Having given the times at which the Sun sets, and is west, on the same day, at a particular place, to find the latitude of the place and the declination.

Example.—Hour-angle when west 4h. 43m. 28s., hour-angle at setting 7h. 35m. 22s.

* **46**. Having given the Sun's declination, and the interval between the times at which he is west, and sets, to find the latitude of the place.

Example.—Declination 20° N., interval 2h. 51m. 54s.

* **47**. Having given the amplitude of the Sun, and the azimuth at six o'clock, to find the latitude of the place and declination of the Sun.

Example. — Amplitude W. 30° 44′ 30″ N., azimuth N. 76° 18′ 45″ W.

* **48**. Having given the Sun's meridian altitude, and his altitude at six o'clock, to find the latitude of the place, and declination of the Sun.

Example.—Meridian altitude 62°, altitude at six o'clock 14° 43′ 30″.

* **49**. Having given the Sun's meridian altitude, and the hour-angle when rising, to find the latitude of the place.

Example.—Meridian altitude 56°, hour-angle at rising 7h.

* **50**. Having given the interval between the times at which the Sun bears west, and sets, at a place whose latitude is known, to find the declination of the Sun.

Example.—Latitude 48° N., interval 2h. 51m. 54s.

* **51**. At a given place, to find the greatest azimuth of a heavenly body whose declination is greater than the latitude of the place; to find also the time and altitude on a given day when the heavenly body will have the greatest azimuth, and when, consequently, it will appear to move perpendicularly to the horizon.

Example.—In latitude 20° N. when the Sun's declination is 23° 28′ N., required the time and altitude when its azimuth is the greatest, and also its greatest azimuth.

* **52**. In latitude 20° N. when the declination is 23° 28′ N., required the time when the Sun will appear stationary in azimuth, the period during which the shadow will move in a contrary direction, and the number of degrees through which it will appear to go back.

* **53**. When the Sun's declination was 10° 15′ N., and that of the Moon was 12° 46′ S., they were observed to rise at the same time; required the latitude of the place and the time of the observation, the right ascension of the Sun being 1h. 53m. 42s. greater than that of the Moon.

* **54**. Find the declination of the Sun when he is in the horizon of Dublin and of Pernambuco at the same instant, the respective latitudes being 53° 21′ N. and 8° 13′ S., and the longitudes 6° 19′ W. and 35° 5′ W.

* **55**. ABCD is a square field, each of whose sides is 100 yards. In the middle of the field stands an obelisk 60 feet high; find the altitude of the Sun when the shadow of the obelisk just reaches the corner of the square.

* **56**. At noon on the shortest day the shadow of a perpendicular stick was seven times as long as its shadow at noon on the longest day; required the latitude, the declination being 23° 28′.

* **57**. Compare the lengths of the shadow of a perpendicular stick at noon in latitude 45° N. on the two days when the Sun's declination is 15° N. and 15° S. respectively.

* **58**. In latitude 33° 30′ N. I observed that my shadow bore to my height the proportion of 5 : 3; required the altitude and hour-angle of the Sun, having given the declination 10° 15′ N.

* **59**. The length of the shadow of a perpendicular object was 4 feet, and its longest when sloping was 5 feet; required the Sun's altitude.

* **60**. The elevation of a cloud was observed to be 20°, and at the same time the Sun's altitude was 22°, the Sun and cloud being in the same vertical plane with the observer, whose distance from the shadow was 400 yards. Find the height of the cloud.

* **61**. In latitude 45° N., the meridian altitude of the Sun was 30°; show that the tangent of one quarter of the length of the day was $\frac{1}{\sqrt[4]{3}}$.

* **62**. At a certain place the Sun rose at 7h. A.M., apparent time, and its meridian zenith distance was twice the latitude; required the latitude.

* **63**. In latitude 45° N. the Sun rose at 4h. A.M., apparent time; show that the tangent of the meridian altitude was 3.

* **64**. In latitude 50° N. when the Sun's declination is 5° 38′ N., required the time it will take to rise out of the horizon, its semidiameter being 16′.

* **65**. Required the time the Sun's semidiameter will take to pass the meridian, the declination being 23° 4′ and semidiameter 16′ 17″.

* **66**. In latitude 45°, required the difference in the lengths of the longest and shortest days.

* **67**. In what latitude will the difference between the longest and shortest days be just 6 hours?

* **68**. At a certain place, when the Sun's declination was 10° N. it rose an hour later than when it was 20° N.; required the latitude.

* **69**. In what latitude will the shortest day be just one-third the longest?

* **70**. At a certain place, when the Sun's declination was d, the length of day was 13h. 38m., and when the declination was $2d$ the length of day was 15h. 26m.; find the latitude of the place and the declination of the Sun.

ANSWERS TO THE EXAMPLES.

I. (page 178).

1.	2·394452.	2.	3·169086.	3.	1·147985.	4.	3·622891.
5.	5·082800.	6.	1·692406.	7.	$\bar{2}$·602060.	8.	$\bar{4}$·698970.
9.	$\bar{1}$·876414.	10.	1·954435.	11.	3·000003.	12.	$\bar{3}$·864866.

II. (page 179).

1.	248.	2.	2480000.	3.	260·418.	4.	1·6514.
5.	204·221.	6.	121004·19.	7.	·024.	8.	·000035.
9.	·6945.	10	·00000075.	11.	·004.	12.	·0248.

III. (page 180).

1.

	L sin.	L tan.	L cosec.
(1)	9·246845	9·253720	10·753155
(2)	9·516536	9·541332	10·483464
(3)	9·875079	10·054613	10·124921
(4)	9·943532	10·263631	10·056468

2. 9·413054 and 9·932357. 3. 9·989363 and 9·958137.

IV. (page 181).

1. (1) 25° 58′ 30″. (2) 32° 11′ 33″.
(3) 79° 6′ 13″. (4) 17° 7′ 36″.
(5) 98° 18′ 56″.

2. (1) 35° 27′ 33″, 144° 32′ 27″, 215° 27′ 33″, 324° 32′ 27″.
(2) 46° 7′ 47″, 133° 52′ 13″, 226° 7′ 47″, 313° 52′ 13″.
(3) 80° 47′ 53″, 99° 12′ 7″, 260° 47′ 53″, 279° 12′ 7″.
(4) 81° 51′ 59″, 278° 8′ 1″.

V. (page 182).

1. (1) 0211397. (2) 0662951. (3) 1198665.
2. (1) 21° 58′ 31″. (2) 89° 16′ 25″. (3) 4° 56′ 42″.

VI. (page 182).

1.	8064.	2.	9216.	3.	42854.
4.	42854.	5.	·0009072.	6.	·0000002229.

VII. (page 183).

1. (1) 58·96. (2) ·26389. (3) 2·222. (4) 2187·23.
2. (1) 25·26. (2) 1295·71.

VIII. (page 185).

1.	1953·127.	2.	11078·5.	3.	1507·82
4.	11·989.	5.	247742·3.	6.	·00032.
7.	·000070494.	8.	·0063241.	9.	3·79195.
10.	19·105.	11.	23·1116.	12.	1·071776.
13.	·304959.	14.	1·47923.	15.	·10772.
16.	·2321.	17.	·09644.	18.	·036342.
19.	10·544.	20.	·272016.	21.	30·586.
22.	1717740.	23.	·000000000003741.	24.	·695883.
25.	·0004572.	26.	·85475.	27.	1·8593.
28.	·0016.				

IX. (page 186).

1. (1) $\log. x = 2 \log. a + \log. b + \log. c + 2 \log. d$.
   (2) $\log. x = 2 \log. a + \log. b + \frac{1}{3} \log. c - 1$.
   (3) $\log. x = \frac{1}{2} (\log. a + \log. b) - \log. c - 2 \log. d$.
   (4) $\log. x = \frac{1}{3} (\log. a + \log. b + 4 \log. c) - \frac{1}{4} \log. d$.
2. (1) 110592. (2) ·07509. (3) ·00000675.
   (4) 107·124. (5) 54·95. (6) 1747·6.
   (7) 94·795.
3. (1) $x = 2·4101$. (2) $x = 1·3634$. (3) $x = ·1587$.
   (4) $x = ·6057$. (5) $x = ·06183$. (6) $x = 1·0054$.
   (7) $x = ·1275$. (8) $x = ·6827$.
4. (1) $x = \frac{\log. b}{\log. a}$. (2) $x = \frac{\log. c - \log. b}{m \log. a - n \log. b}$.
   (3) $x = \frac{n \log. b}{m \log. b - \log. a - r \log. c}$.
5. (1) $-1·274$. (2) ·7153. (3) $\sqrt{x} = -·5$.

X. (page 188).

1. $x = 144·5$. 2. $x = ·074764$. 3. $x = -7·4382$. 4. $x = 19378·93$.

XI. (page 191).

1. (1) $c = 27·31$, $A = 52° 37' 30''$, $C = 37° 22' 30''$.
   (2) $b = 506·9$, $A = 49° 14' 15''$, $C = 40° 45' 45''$.
   (3) $c = 4264$, $A = 56° 29' 15''$, $B = 33° 30' 45''$.
   (4) $a = ·1286$, $c = ·1532$, $C = 50°$.
   (5) $a = ·04767$, $b = ·06223$, $A = 50°$.
   (6) $a = 2132·1$, $B = 56° 29' 15''$, $C = 33° 30' 45''$.
2. 195 feet. 3. 238·5 feet. 4. 499·7 feet. 5. 65·8 feet.
6. 3·14 feet. 7. 24 miles. 8. 368·4 feet. 9. At 8^h p.m.

XII. (page 194).

1. A = 89° 45′ 45″, B = 35° 12′ 0″, C = 55° 2′ 15″.
2. A = 54° 8′ 15″, B = 82° 58′ 0″, C = 42° 53′ 45″.
3. A = 56° 4′ 0″, B = 48° 31′ 0″, C = 75° 25′ 0″.
4. A = 29° 44′ 0″, B = 115° 36′ 30″, C = 34° 39′ 30″.
5. A = 10° 58′ 0″, B = 107° 58′ 45″, C = 61° 3′ 15″.
6. A = 107° 46′ 15″, B = 53° 7′ 30″, C = 19° 6′ 15″.
7. A = 28° 14′ 15″, B = 45° 12′ 30″, C = 106° 33′ 15″.
8. A = 20° 11′ 30″, B = 48° 19′ 15″, C = 111° 29′ 15″.

XIII. (page 197).

1. (1) $c = 316{\cdot}3$, B = 36° 6′ 30″, C = 102° 34′ 15″.
 (2) $b = 22{\cdot}68$, B = 105° 52′ 30″, C = 27° 7′.
 (3) $b = 73{\cdot}98$, B = 49°, C = 29° 19′.
 (4) $b = 3$, $c = 4$, C = 88° 2′ 30″.
 (5) $c = 1{\cdot}013$, A = 28° 13′, C = 106° 37′.
 (6) $a = 279{\cdot}7$, $b = 243$, C = 91° 43′.
 (7) $b = 265{\cdot}8$, $c = 207{\cdot}6$, C = 48°.
 (8) A = 54° 2′ 15″ or 125° 57′ 45″, C = 84° 47′ 45″ or 12° 52′ 15″, $c = 219{\cdot}3$ or $49{\cdot}06$.
 (9) B = 80° 39′ 45″ or 99° 20′ 15″, C = 43° 7′ 30″ or 24° 27′, $c = 2136{\cdot}7$ or $1293{\cdot}7$.
2. Bearing of lighthouse from fort, S. 50° 24′ W.
3. 4·59 miles.
4. 3h. 20m. nearly.
5. 8h. 32m. nearly.
6. 2·298 miles.

XIV. (page 199).

1. (1) B = 52° 54′ 45″, C = 60° 44′ 45″, $a = 73{\cdot}5$.
 (2) A = 54° 8′ 15″, B = 82° 57′ 45″, $c = 430$.
 (3) B = 29° 7′ 45″, C = 45° 24′ 45″, $a = 106{\cdot}92$.
 (4) A = 62° 51′ 30″, B = 41° 52′, $c = {\cdot}0391$.
2. 108·52 miles.
3. 1,879 feet.
4. 1,012 yards.
5. S. 11° 32′ E., 1·65 mile per hour.

XV. (page 201).

1. (1) 80627 square yards. (2) 2288·3 square feet.
 (3) 3714 square feet. (4) ·5902 square feet.
 (5) 2362·5 square feet.
2. 536·23 acres.
3. 42·9 feet.
4. 27° 24′ 30″, 102° 37′ 45″, 49° 57′ 45″.
5. 104·4 feet.

XVI. (page 202).

1. (1) 3353 square feet. (2) 1540280 square feet.
2. ·688 square inch.
3. 1·7604 acres.
4. 9768·7 square yards.

XVII. (page 204).

1. $A = 59° \ 2' \ 15''$, $B = 74° \ 54' \ 0''$, $C = 71° \ 18' \ 30''$.
2. $A = 115° \ 39' \ 0''$, $B = 99° \ 21' \ 30''$, $C = 75° \ 31' \ 45''$.
3. $A = 81° \ 24' \ 45''$, $B = 61° \ 31' \ 45''$, $C = 102° \ 59' \ 30''$.
4. $A = 147° \ 3' \ 0''$, $B = 113° \ 28' \ 0''$, $C = 90° \ 0' \ 15''$.
5. $A = 47° \ 55' \ 45''$, $B = 61° \ 50' \ 30''$, $C = 89° \ 59' \ 30''$.

XVIII. (page 206).

1. (1) $a = 96° \ 10' \ 2''$. (2) $a = 46° \ 19' \ 38''$.
   (3) $c = 87° \ 0' \ 48''$. (4) $a = 23° \ 57' \ 10''$.
   (5) $c = 100° \ 9' \ 33''$. (6) $c = 105° \ 0' \ 11''$.
2. (1) $B = 49° \ 30' \ 15''$, $C = 100° \ 43' \ 45''$.
   (2) $A = 111° \ 18' \ 15''$, $B = 80° \ 5' \ 45''$.

XIX. (page 209).

1. (1) $B = 56° \ 57' \ 15''$. (2) $B = 36° \ 51' \ 15''$ or $143° \ 8' \ 45''$.
2. $B = 43° \ 44'$, $c = 101° \ 1' \ 30''$, $C = 108° \ 1' \ 30''$.

XX. (page 210).

(1) $b = 78° \ 13' \ 0''$. (2) $b = 77° \ 16' \ 15''$ or $102° \ 43' \ 45''$.

XXI. (page 212).

(1) $a = 87° \ 10' \ 30''$, $b = 62° \ 36' \ 45''$, $c = 100° \ 10' \ 15''$.
(2) $a = 79° \ 26' \ 15''$, $b = 84° \ 49' \ 0''$, $c = 97° \ 26' \ 0''$.
(3) $a = 66° \ 48' \ 0''$, $b = 101° \ 25' \ 0''$, $c = 123° \ 23' \ 30''$.

XXII. (page 216).

(1) $a = 94° \ 57' \ 15''$, $B = 60° \ 32' \ 45''$, $C = 98° \ 41' \ 45''$.
(2) $a = 90° \ 30' \ 0''$, $b = 100° \ 0' \ 45''$, $c = 87° \ 7' \ 15''$.
(3) $a = 51° \ 46' \ 15''$, $b = 26° \ 23' \ 30''$, $C = 66° \ 59' \ 30''$.
(4) $B = 101° \ 31' \ 15''$, $C = 111° \ 58' \ 0''$, $c = 112° \ 26' \ 45''$.
(5) $b = 74° \ 5' \ 45''$, $c = 132° \ 39' \ 30''$, $C = 131° \ 32' \ 45''$.
(6) $a = 91° \ 47' \ 15''$, $b = 91° \ 4' \ 15''$, $C = 53° \ 15' \ 0''$.
(7) $c = 84° \ 24' \ 0''$, $A = 119° \ 50' \ 15''$, $B = 99° \ 39' \ 30''$.
(8) $a = 54° \ 28' \ 0''$ or $125° \ 32' \ 0''$, $c = 23° \ 2' \ 30''$ or $156° \ 57' \ 30''$, $C = 28° \ 44' \ 45''$ or $151° \ 15' \ 15''$.

XXIII. (page 218).

(1) $b = 78° \ 14' \ 30''$, $B = 74° \ 36' \ 30''$, $C = 49° \ 8' \ 0''$.
(2) $b = 47° \ 44' \ 30''$, $A = 107° \ 10' \ 15''$, $C = 65° \ 19' \ 15''$.
(3) $b = 82° \ 26' \ 0''$, $c = 50° \ 27' \ 0''$, $A = 96° \ 17' \ 45''$.
(4) $c = 108° \ 0' \ 0''$, $B = 45° \ 0' \ 0''$, $C = 114° \ 40' \ 45''$.
(5) $A = 101° \ 42' \ 30''$, $B = 72° \ 20' \ 15''$, $C = 48° \ 1' \ 30''$.
(6) $A = 59° \ 41' \ 45''$, $B = 78° \ 51' \ 0''$, $C = 95° \ 36' \ 0''$.

Miscellaneous Examples in Plane Trigonometry.

1. 190·4 feet.
2. 205 feet.
3. $3^{m.}$ $21{\cdot}6^{s.}$
4. 51·76 feet.
5. 75 feet.
6. 49·44 feet.
7. 23·6 yards.
8. 1100·1 yards.
9. 9·478 miles and 9·52 miles.
10. 14·4 feet.
11. 145·9 yards.
12. 942·7 yards.
13. 162 feet; 24° 50′.
14. 5147·9 feet.
15. 219·8 feet.
16. 66·4 feet.
17. 1174·4 yards.
18. 1290 yards.
19. 46·67 yards.
20. From A 536 yards, from B 500 yards.
21. 17° 47′ 45″.
22. 511·3 yards.
23. 210·4 feet.
24. 314·2 feet.
25. 529·5 yards.
26. 278·7 feet.
27. 5422 yards.
28. 8·5 and 10·8 miles.
29. S. 87° 40′ W. 42·3 miles.
30. S. 58° 40′ W. 9·71 miles.
31. 192 yards.
32. N. 20° 32′ E.
33. 9° 28′; 2·9 times the height.
34. 50 feet.
35. 18° 35′ 15″.
36. 633·4 feet.
37. 4694 yards.
38. 4 or $\sqrt{52}$ feet.
39. 1236·1 yards.
40. 20·5 yards.
41. 1·402 miles.
42. 401·4 feet.
43. 187·5 feet.
44. 1·268, 2·196, 2·536.
45. 19·8, 35·69, 44·51 yards.
46. 5·072, 8·784, 10·144 yards.
47. 369 feet and 231 feet.
48. 10 feet and 6 feet.
49. 150 feet.
50. $a = 103{\cdot}7$ feet, $b = 90{\cdot}7$ feet, A = 79° 14′, B = 59° 14′.

Miscellaneous Examples in Spherical Trigonometry.

1. 40° N.
2. 15° N.
3. 5° N.
4. 50° S.
5. 70°.
6. 10° N.
7. 33° S.
8. 45° N. or S., according as the declination is N. or S.
9. 55° N. or S.
10. 45° N. or S.
11. 70° and 10°.
12. Dec. 50° 48′ N.; alt. 11° 36′.
13. 66° 32′ N. or S.
14. N. 111° 51′ W.; 2h. 57m. 16s.
15. 2h. 57m. 16s.
16. 43° 49′.
17. N. 44° 12′ E.
18. 59° 4′ N.
19. 71° 31′ N.
20. 43° 9′ N.
21. At 4h. 37m. 4s. P.M.; 30° 55′.
22. 59m. 56s.
23. 69° 32′ N.
24. Alt. 17° 58′; Az. N. 74° 39′ E.
25. 6h. 31m. 7s.
26. 64° 29′ N.
27. E. 30° 4′ N.; 15h. 13m. 22s.
28. E. 19° 5′ N.

29. 56° 33′.
30. Dec. 21° 4′ N.; long. 64° 33′.
31. Lat. 4° 14′ 45″ N.; long. 48° 37′ 30″.
32. Dec. 25° 50′ N.; R.A. 16h. 14m. 3s.
33. 449·7 miles.
34. 5,950 miles.
35. S. 43° 38′ W.
36. 72° 41′ S.
37. 17° 24′.
38. N. 75° 5′ W.
39. 19° 55′ 30″ N.
40. 29° 42′ N.
41. 58° 31′ N.
42. Lat. 48°; Dec. 20°.
43. Lat. 42° or 48° N.; Dec. 22° 20′ or 20° N.
44. Lat. 48°; Dec. 20°.
45. Lat. 48°; Dec. 20°.
46. 42° or 48°.
47. Lat. 48° N.; Dec. 20° N.
48. Lat. 48° 0′ 15″; Dec. 20°.
49. 47° 24′.
50. 20° N.
51. Azimuth N. 77° 28′ E.; time 9h. 47m. 53s. A.M.; altitude, 59° 11′ 30″.
52. Time 9h. 47m. 53s. A.M.; period 4h. 24m. 15s.; shadow went back through 12° 32′ 30″.
53. Lat. 50° 18′ 30″ N.; time 5h. 9m. 40s. A.M.
54. 18° 6′.
55. 15° 47′ 30″.
56. 38° 27′ 45″.
57. As 1 to 3.
58. Alt. 30° 58′; hour angle 3h. 58m. 3s.
59. 36° 52′ 15″.
60. 1,468 yards.
62. 26° 58′.
64. 3m. 21s.
65. 1m. 11s.
66. 6h. 51m. 40s.
67. 41° 24′.
68. 52° 27′ N.
69. 58° 27′.
70. Lat. 54° 17′; $d = 8° 40′ 30″$.

APPENDIX

A COLLECTION OF EXAMPLES

SELECTED FROM

EXAMINATION PAPERS SET AT THE ROYAL NAVAL COLLEGE

BETWEEN THE YEARS 1880—1893

WITH ANSWERS

APPENDIX.

A.—PART I. CAP. I. II.

N.B.—*In the following examples, unless otherwise stated,* π *is taken as* $\frac{22}{7}$.

1. If on a map a square inch represents 10 acres, how many yards are represented by the diagonal of a square inch?

2. Find the distance at which a person looking towards the Sun must hold a coin whose diameter is half an inch in order that it may just hide the Sun, assuming that the angle subtended by the Sun's diameter is 32′.

3. An angle measuring 1·25 is subtended by an arc of 16 feet: calculate the radius.

4. Determine the number of degrees in the angle subtended at the centre of a circle, the radius of which is 10 feet, by an arc the length of which is 9 inches.

5. Find the measure, both in degrees and circular measure, of the angle between two consecutive spokes of a wheel of 14 spokes.

6. If D, θ represent the same angle, as measured in degrees and circular measure respectively, and π be the circular measure of two right angles, show that

$$\frac{D}{90} = \frac{2\theta}{\pi}.$$

7. (*a*) Find the radius of a planet on which the arc joining two places that have the same longitude, but latitudes differing by 60°, is 2,000 miles long.

(*b*) Find the diameter of a globe on which an arc of 21° of a meridian measures 5ft. 6in.

8. How large a mark on a target 1,000 yards off will subtend an angle of one second at the eye?

9. It has been found from the transit of Venus in 1882 that the Earth's semidiameter subtends an angle of $8''\cdot82$ at the Sun.

Find the Sun's distance, the Earth's radius being 3,963 miles.

10. Assuming the Moon's distance from the Earth to be equal to 60 times the Earth's radius, express in degrees and in circular measure the approximate value of the angle subtended at the Moon by the Earth's radius.

11. The measure of the same angle is given by one person as 15, and by another as $\cdot1\dot{6}$. Taking it for granted that the first person means 15 degrees, find the unit employed by the second person.

12. Find the number of degrees in the unit of angular measurement when the right angle is measured by $\frac{2}{\pi}$.

13. The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle of the same radius. Find the number of degrees, minutes, and seconds in the angle of the sector.

14. If an angle subtended by an arc equal to π times the radius be taken as the unit, what number will measure an angle of 45°?

15. One angle of a triangle is 30°, and the circular measure of another is $\frac{5}{4}$: find the circular measure of the third angle.

16. Find the angle whose circular measure is equal to π times the square root of the number of right angles in the angle.

17. In sailing three-quarters of a mile a ship changes the direction of her course from S.E. to E.S.E.: find the radius of the circular arc she describes.

18. The arc of a certain sector of a circle is equal in length to the sum of its bounding radii: find the number of degrees, minutes, and seconds in the angle between these radii.

19. In a circle of radius unity a certain arc subtends at the centre an angle whose circular measure is A. In a second

circle an arc of the same length subtends an angle of A°: find the radius of the latter.

20. Show that the supplement of half the complement of an angle is greater than the complement of half its supplement by the angle of a regular octagon.

21. The circular measure of the difference between the vertical angle and either of the angles at the base of an isosceles triangle is $\frac{\pi}{4}$, the vertical angle being the smaller. Express the angles in degrees.

22. The four angles of a quadrilateral figure are in arithmetical progression, and the greatest is to the least as 17 to 3. Express each angle in degrees.

23. The angle subtended at the centre by a certain chord is such that the square root of its cosine is equal to the ratio of the chord to the radius of the circle. Express the angle both in degrees and circular measure.

24. Show with the aid of logarithmic tables that log. cos. θ = log. (circular measure of θ) very nearly, when $\theta = 42^\circ\ 18'\ 5''$ and $\pi = 3{\cdot}14159$.

B.—Part I. Cap. VIII.–XI.

Prove the identities:—

25. $(\sin. A + \cos. A)(\sin.^3 A + \cos.^3 A)$
$- (\sin. A - \cos. A)(\sin.^3 A - \cos.^3 A) = 2 \sin. A \cos. A.$

26. $1 + \tan.^6 A = \sec.^4 A (\sec.^2 A - 3 \sin.^2 A).$

27. $\sec.^2 A \text{ vers. } (90^\circ - A) \text{ vers. } (90^\circ + A) = 1.$

28. $(\tan. A + \cot. A - 1)(\sin. A + \cos. A) = \tan. A \sin. A + \cot. A \cos. A.$

29. $1 + \cos.^4 A \operatorname{cosec.}^2 A + \sin.^4 A \sec.^2 A = \cot.^2 A + \tan.^2 A.$

30. $(1 + \cos. A \operatorname{cosec.}^2 A + \cos.^2 A \operatorname{cosec.}^2 A) \text{ vers. } A = 1.$

31. $\{(\cos. A + \sin. A)^2 - 1\}(1 - \tan.^2 A)$
$= 2(\cos.^2 A - \sin.^2 A) \tan. A.$

32. $(\sin. A - \operatorname{cosec.} A)^2 + (\cos. A + \sec. A)^2$
$= 1 + \sec.^2 A \operatorname{cosec.}^2 A.$

33. $$\frac{1 - \sin. A \cos. A}{\cos. A (\sec. A - \operatorname{cosec.} A)} \times \frac{\sin.^2 A - \cos.^2 A}{\sin.^3 A + \cos.^3 A} = \sin. A.$$

34. $\sin. 4A = 4 \sin. A \cos.^3 A - 4 \cos. A \sin.^3 A.$

35. $\sin. 5A = 16 \sin.^5 A - 20 \sin.^3 A + 5 \sin. A.$

36. $\dfrac{1 + 2 \cos. A}{1 + \cos. A + \cos. 2A} = \sec. A.$

37. $2 \sin. 2A + \sin. 4A = 4 \sin. 2A \cos.^2 A.$

38. $(\cos. 2A - \cos. 4A) \cos. 3A = (\sin. 4A - \sin. 2A) \sin. 3A.$

39. $\tan. 2A - \tan. A = \dfrac{2 \sin. A}{\cos. A + \cos. 3A}.$

40. $\sin. 3A + \sin. 5A = 8 \sin. A \cos.^2 A \cos. 2A.$

41. $\sin. A + \sin. 3A + \sin. 5A + \sin. 7A$
$= 16 \sin. A \cos.^2 A \cos.^2 2A.$

42. $\sin. 2A + \sin. 4A + \sin. 6A - \sin. 12A$
$= 4 \sin. 3A \sin. 4A \sin. 5A.$

43. $\dfrac{\cot. A + \cot. 4A}{\cot. 2A + \cot. 3A} = 1 + \dfrac{1}{2} \sec. 2A.$

44. $\dfrac{1}{3} (\cos.^6 A + \sin.^6 A) - \dfrac{1}{4} (\cos.^4 A - \sin.^4 A)^2 = \dfrac{1}{12}.$

45. $(\cos. A + \sin. A) (\cos. 2A + \sin. 2A) = \cos. A + \sin. 3A.$

46. $\dfrac{\tan. A + \cot. A + 2}{\tan. A + \cot. A - 2} = \dfrac{\sin.^2 \left(\dfrac{\pi}{4} + A\right)}{\sin.^2 \left(\dfrac{\pi}{4} - A\right)}.$

47. $\sin. A \sin. 2A + 2 \cos. A \cos. 2A = 2 \cos.^3 A.$

48. $\sec.^2 A - \tan.^2 A (1 + 2 \cos. 2A)^2 = (1 - 2 \cos. 2A)^2.$

49. $1 + (\cos.^2 A - \sin.^2 A)^3 = 2 \cos.^2 A (\cos.^4 A + 3 \sin.^4 A).$

50. $\cos. 6A = 16 (\cos.^6 A - \sin.^6 A) - 15 \cos. 2A.$

51. $\tan. 3A - \tan. 2A - \tan. A = \tan. A \tan. 2A \tan. 3A.$

52. $\sin. A + \sin. 2A + \sin. 3A = 4 \sin. \dfrac{3}{2} A \cos. A \cos. \dfrac{1}{2} A.$

53. $\tan. \dfrac{\pi + A}{4} + \tan. \dfrac{\pi - A}{4} = 2 \sec. \dfrac{A}{2}.$

54. $(\sin. 2A + 2 \sin. A)^2 + (\cos. 2A + 2 \cos. A + 1)^2 = 16 \cos.^4 \dfrac{A}{2}.$

55. $\cos.^2 \dfrac{A}{2} (1 - 2 \cos. A)^2 + \sin.^2 \dfrac{A}{2} (1 + 2 \cos. A)^2 = 1.$

56. $\tan. A + \cot. \dfrac{A}{2} = \dfrac{\operatorname{cosec.}^2 \dfrac{A}{2}}{\cot. \dfrac{A}{2} - \tan. \dfrac{A}{2}}.$

57. $\cos. A - \tan. \frac{A}{2} \sin. A = \cos. 2A + \tan. \frac{A}{2} \sin. 2A.$

58. $\cos. A + \cos. 2A + \cos. 3A + \cos. 4A$

$$= 4 \cos. \frac{A}{2} \cos. A \cos. \frac{5A}{2}.$$

59. $\log. (\cos. A) + \log. (\cos. 2A) + \log. (\cos. 4A)$

$$= \log. (\sin. 8A) - 3 \log. 2 - \log. (\sin. A).$$

60. $\tan. (A + B) = \dfrac{\sin.^2 A - \sin^2 B}{\sin. A \cos. A - \sin. B \cos. B}.$

61. $\sin. (A - B) \cos. (A + B) + \sin. (B - C) \cos. (B + C) + \sin. (C - D) \cos. (C + D) + \sin (D - A) \cos. (D + A) = 0.$

62. $4 \sin. 5A \sin. 5B \sin. 5 (A + B) = \sin. 10 A + \sin. 10 B - \sin. 10 (A + B).$

63. $\sin. 2A + \sin. 2B + \sin. 2C = \sin. 2 (A + B + C) + 4 \sin. (A + B) \sin. (B + C) \sin. (C + A).$

64. $\sin. (A + B) \sin. (B + C + D) = \sin. A \sin. (C + D) + \sin. B \sin. (A + B + C + D).$

65. $\dfrac{2 (\sin. A + \sin. B)}{\left\{\cos. \frac{A + B}{2} + \cos. \frac{A - B}{2}\right\}^2}$

$$= \left(\tan. \frac{A}{2} + \tan. \frac{B}{2}\right) \left(1 + \tan. \frac{A}{2} \tan. \frac{B}{2}\right).$$

66. $\cot. (A + B) + \cot. (A - B) = \dfrac{\sin. 2A}{\cos.^2 B - \cos.^2 A}.$

67. $2 \{1 - \cos. (A - B) \cos. A \cos. B\} = \sin.^2 A + \sin.^2 B + \sin.^2 (A - B).$

68. $\sqrt{\dfrac{\sin. 45^\circ - \sin. 30^\circ}{\sin. 45^\circ + \sin. 30^\circ}} = \sec. 45^\circ - \tan. 45^\circ.$

69. $\cos. (A + 15^\circ) + \sin. (A - 15^\circ) = \sin. (A + 45^\circ).$

70. $\sin. \dfrac{3\pi}{16} + \sin. \dfrac{7\pi}{16} + \sin. \dfrac{11\pi}{16} + \sin. \dfrac{15\pi}{16}$

$$= 4 \cos. \frac{\pi}{4} \cos. \frac{\pi}{8} \cos. \frac{\pi}{16}.$$

71. $\cos. 75^\circ \times \cos. 15^\circ = \dfrac{1}{4}.$

72. $\tan. 60^\circ - \tan. 165^\circ = 2.$

73. $\sin.^4 15^\circ + \cos.^4 15^\circ = \dfrac{7}{8}.$

74. $\tan.^2 72° + \tan.^2 36° = 10$

75. $\cos.^2 72° + \cos.^2 36° = \frac{3}{4}$.

76. $\cos.^2 18° + \cos.^2 30° + \cos.^2 54° = 2$.

77. $\sin. 87° - \sin. 59° - \sin. 93° + \sin. 61° = \sin. 1°$.

78. $\tan. 36° - \sqrt{5} \tan. 18° = 0$.

79. (*a*) $4 \sin. 9° = \sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}$.

(*b*) $2 \sin. 11° 15' = \sqrt{\{2 - \sqrt{2 + \sqrt{2}}\}}$.

80. If $\sin. A = \frac{24}{25}$, find the values of $\sin. \frac{A}{2}$.

81. If A lies between 270° and 360°, show that $2 \cos. \frac{A}{2} = -\sqrt{1 - \sin. A} - \sqrt{1 + \sin. A}$.

82. If $\cot. A = 2\sqrt{2}$, find sin. A.

83. If $\sin. A = \frac{7}{25}$, $\sin. B = \frac{20}{29}$, find tan. (A − B).

84. If $\tan. A = \frac{5}{12}$, $\tan. \frac{B}{2} = \frac{3}{4}$, find sin. (A + B).

85. If $\cos. A = \frac{1}{4}(\sqrt{6} + \sqrt{2})$, and $\cos. B = \frac{\sqrt{3}}{2}$, find cos. (A + B).

86. If $\tan. A = \frac{b}{a}$, prove that

$$\sqrt{\frac{a + b}{a - b}} + \sqrt{\frac{a - b}{a + b}} = \frac{2 \cos. A}{\sqrt{\cos. 2A}}.$$

87. If $\sin. (A - B) = \frac{1}{6}$, $3 \tan. A = 5 \tan. B$, find the value of sin. (A + B).

88. If $\cot. A = 5 \cos. B = 3$, show that $2A + B = 90°$.

89. If the sine of an angle in the third quadrant is $-\frac{3}{5}$, find the cosine of half the angle.

90. The cosine of an angle in the second quadrant is $-\frac{3}{\sqrt{130}}$: find the values of the sine and tangent.

91. If $\tan. \theta = \frac{b}{a}$, show that $a \cos. 2\theta + b \sin. 2\theta = a$.

92. If $\cot. A = 2 \tan. (A - B)$, show that $\tan. B = \frac{1 - 3 \cos. 2A}{3 \sin. 2A}$.

C.—Part I. Cap. XII.–XIV.

93. If $\sin.\ \theta + 3\cos.\ \theta = 1$, find $\tan.\ \theta$.

94. If $5\sin.\ \theta + 12\cos.\ \theta = 13$, find $\tan.\ \theta$.

95. If $\tan.\ \theta \tan.\ 3\theta = -\frac{2}{5}$, find $\tan.\ \theta$ and $\tan.\ 3\theta$.

96. If $\sec.\ A - \operatorname{cosec.} A = 2\sqrt{2}$, find $\sin.\ 2A$.

97. Having given that $(\sin.\ \theta + \cos.\ \theta)^2 = \frac{1 + 2\sqrt{2} + \sqrt{3}}{2\sqrt{2}}$, find $\cos.\ 4\theta$.

98. Solve the equations :—

(*a*) $\sin.\ \theta + \cos.\ 2\theta = 1$.

(*b*) $11\cos.\ \theta + 6\sin.^2\theta = 10$.

(*c*) $\tan.\ (45° + \theta) = 1 + \tan.\ \theta$.

(*d*) $2\cos.\ \theta = 3 + 2\sec.\ \theta$.

(*e*) $\frac{1 - \sin.\ \theta}{1 + \sin.\ \theta} = (1 - 2\sin.\ \theta)^2$.

99. Having given that $a\cos.\ \theta + b\sin.\ \theta = c$, and that $\frac{a}{\cos.\ \alpha} = \frac{c}{\cos.\ \beta} = \sqrt{a^2 + b^2}$, prove that $\theta = 2n\pi + \alpha \pm \beta$.

100. Solve the equation :

$$\sin.\ 2x + \sin.\ 30° = \sqrt{2}\sin.\ (x + 45°).$$

Prove the following relations :—

101. $\cot.^{-1}\frac{1}{3} = \cot.^{-1} 3 + \cot.^{-1}\frac{3}{4}$.

102. $\cot.^{-1} 3 + \operatorname{cosec.}^{-1}\sqrt{5} = \frac{\pi}{4}$.

103. $\sin.^{-1}\frac{4}{5} + \sin.^{-1}\frac{8}{17} + \sin.^{-1}\frac{13}{85} = \frac{\pi}{2}$.

104. $\tan.^{-1}\frac{2}{11} + 2\tan.^{-1}\frac{1}{7} = \tan.^{-1}\frac{1}{2}$.

105. $\cot.^{-1}\frac{1}{2} + \cot.^{-1}\frac{1}{3} - \cot.^{-1}\frac{1}{2 + \sqrt{3}} = \frac{\pi}{3}$.

106. $\tan.^{-1}\frac{4}{3} = \frac{1}{2}\tan.^{-1}\left(-\frac{24}{7}\right) = \frac{1}{3}\tan.^{-1}\left(-\frac{44}{117}\right)$.

107. $\tan.^{-1}\frac{2a - b}{b\sqrt{3}} + \tan.^{-1}\frac{2b - a}{a\sqrt{3}} = \frac{\pi}{3}$.

108. $\cot.\left(\cot.^{-1}\frac{3}{4}+\cot.^{-1}\frac{1}{7}\right)=-1.$

109. $2\tan.^{-1}\sqrt{\frac{a-b}{a+b}}\tan.\frac{x}{2}=\cos.^{-1}\frac{b+a\cos.x}{a+b\cos.x}.$

110. Find x from the equation :

$$\tan.^{-1}(x-1)+\tan.^{-1}(2-x)=2\tan.^{-1}\sqrt{3x-x^2-2}.$$

111. Having given $\log._{10}2 = \cdot301030$, $\log._{10}3=\cdot477121$, find $\log._2 3$.

112. Using the values of the logarithms given above, solve the equations :

$$4^x = 500\,;\ (16\cdot2)^x = 900.$$

113. Solve also the equation :

$$(25)^{8-2x} = 2^{x+8}.$$

114. Prove that $(1+2\log._a r)\log._{(ar^2)} a = 1$.

115. Prove that if $\log. a$, $\log._y b$ $\log._z c$ are in arithmetic progression, with a common difference unity, then $xz = y^m$,

$$\text{where } m=\frac{\log. a}{\log. b-\log. y}+\frac{\log. c}{\log. b+\log. y}.$$

116. Show that $\log._r\{\log._a b^r \log._b c^{r^2}\log._c a^{r^3}\} = 6$.

D.—Part I. Cap. XVI.–XIX.

In any plane triangle prove the following relations:—

117. $\cot. B - \cot. A = \frac{a^2-b^2}{ab}\operatorname{cosec.} C.$

118. $\cos.^2\frac{C}{2}-\cos.^2\left(A+\frac{C}{2}\right)=\sin. A\sin. B.$

119. $\frac{\cos.\frac{A}{2}}{\sqrt{a(s-a)}}=\frac{\cos.\frac{B}{2}}{\sqrt{b(s-b)}}=\frac{\cos.\frac{C}{2}}{\sqrt{c(s-c)}}.$

120. $\sin. 2A\sin. 2B(\tan. A\tan. B\tan. C-\tan. C)=$ $4\sin. A\sin. B\sin. C.$

121. $\frac{1}{a}\cos.^2\frac{A}{2}+\frac{1}{b}\cos.^2\frac{B}{2}+\frac{1}{c}\cos.^2\frac{C}{2}=\frac{(a+b+c)^2}{4abc}.$

122. $\frac{a\sin. A+b\sin. B+c\sin. C}{ab\cos. C+ac\cos. B+bc\cos. A}=\frac{2\sin. A}{a}.$

123. $$\frac{b \cos. B - c \cos. C}{bc \cos. A} + \frac{c \cos. C - a \cos. A}{ca \cos. B} + \frac{a \cos. A - b \cos. B}{ab \cos. C} = 0.$$

124. Show that in any plane triangle

(*a*) $$\text{Area} = \frac{a^2}{2} \frac{\sin. B \sin. C}{\sin. (B + C)}$$

(*b*) $$\text{Area} = \frac{1}{2} \frac{a^2}{\cot. B + \cot. C}$$

(*c*) $$\text{Area} = \frac{1}{2} (b^2 + c^2) \frac{a \sin. B \sin. C}{b \sin. B + c \sin. C}$$

(*d*) $$\text{Area} = \frac{a^2 + b^2 + c^2}{4 (\cot. A + \cot. B + \cot. C)}.$$

125. In a plane triangle ABC, in which $A = 90°$, prove that

$$1 + \tan. \frac{B - C}{2} : 1 - \tan. \frac{B - C}{2} :: b : c.$$

126. If C be a right angle, prove that

$$\tan.^2 \frac{A}{2} = \frac{c - b}{c + b}.$$

127. Show also, if C be a right angle, that

$$\tan. \frac{A}{2} + \tan. \frac{B}{2} = \frac{2c}{a + b + c}.$$

128. A plane triangle ABC is such that

$$\frac{\cos. 2A - \cos. 2B}{\sin. 2A} = \tan. B - \tan. A;$$

show that it is either isosceles or right-angled.

129. In any triangle prove that

$$a \cos. A + b \cos. B + c \cos. C = \frac{8 S^2}{abc}.$$

130. The bisector of the angle A of a triangle ABC meets the opposite side in D. Prove that

$$\tan. ADB = \frac{b + c}{b - c} \tan. \frac{A}{2}.$$

131. If x, y be the lengths of the two diagonals of a quadrilateral figure, and θ the angle between them, show that

$$\text{Area} = \tfrac{1}{2} x y \sin. \theta.$$

132. In any plane triangle ABC, if

$$x + \frac{1}{x} = \cos. A, \; y + \frac{1}{y} = 2 \cos. B,$$

prove that one of the values of $bx + \frac{a}{y}$ is c.

133. If the sines of the angles of a triangle are in the ratios of 13 : 14 : 15, prove that the cosines are in the ratios of 39 : 33 : 25.

134. The sides of a triangle are $2xy + x^2$, $x^2 + xy + y^2$, $x^2 - y^2$. Show that the angles opposite are in arithmetical progression, the common difference being

$$2 \tan.^{-1} \frac{y\sqrt{3}}{2x+y}.$$

135. If from any angle of an equilateral triangle any straight line be drawn, and from the other two angles perpendiculars p, q, be drawn to the straight line, prove that the area of the triangle is $\frac{p^2 + pq + q^2}{\sqrt{3}}$.

136. If the cosines of the angles A, B, C of a plane triangle are in arithmetical progression, show that $s - a$, $s - b$, $s - c$ are in harmonical progression.

137. If D and E are the points of trisection of the side BC of a plane triangle ABC, prove that

$$9 \cot. ADB \cot. AEB = 2 (\cot.^2 B + \cot.^2 C) - 5 \cot. B \cot. C.$$

138. If from the angular points of a plane triangle ABC lines be drawn outwards, inclined at an angle of 30° to the sides, show that the vertices of the isosceles triangles so formed will be at equal distances from each other.

139. If AD, BE, CF are the perpendiculars drawn from the angular points of a plane triangle ABC upon the opposite sides, prove that the area of the triangle DEF to that of ABC is as 2 cos. A. cos. B. cos. C to 1.

140. If a straight line of length p bisect the angle A of a triangle ABC, and divide the base into two parts of lengths m and n, prove that $p^2 = bc - mn$.

141. If ABC be a plane triangle, and another triangle be

constructed having c, $(a + b)$ sin. $\frac{C}{2}$, $(a - b)$ cos. $\frac{C}{2}$ for the respective lengths of its sides, show that one of its angles is a right angle.

142. An equilateral triangle is described having its angular points on three given parallel straight lines, of which the outer ones are at distances a and b from the middle one.

Show that the side of the triangle

$$= \frac{2}{3}\sqrt{3(a^2 + ab + b^2)}.$$

143. In the triangle ABC, D is a point in BC such that BD $= 2$ CD; show that

$$\text{AD} = \frac{1}{3}\sqrt{6b^2 + 3c^2 - 2a^2}.$$

144. The sides of a triangle are in arithmetical progression, and its area is four-fifths that of an equilateral triangle of the same perimeter; show that the sides of the triangle are as $7 : 10 : 13$.

145. The sides of a plane triangle are in the proportion of a, b, $\sqrt{a^2 + b^2 + ab}$; show that the greatest angle $= 120°$.

146. Standing on a horizontal plane, I observe that the elevation of a hill due north is α, and after walking a yards due east, its elevation is β. Show that its height above the horizontal plane is

$$\frac{a \text{ sin. } \alpha \text{ sin. } \beta}{\sqrt{\text{sin. } (\alpha + \beta) \text{ sin. } (\alpha - \beta)}}.$$

147. At the top P of a tower of height h, the angles of depression of two objects in the horizontal plane upon which the tower stands are $45° - \alpha$ and $45° + \alpha$ respectively, P, A and B being in the same vertical plane. Show that AB $=$ $2h$ tan. 2α.

148. A flag-staff standing in the centre of a circular pond, whose radius is equal to the height of the staff, is observed to subtend an angle of 45°, from the top of a column 20 feet high, whose base is 12 feet from the edge of the pond. Show that the height of the staff is 68 feet.

149. Standing on an eminence a feet above the level of a lake, I observe the elevations of the top and bottom of a tower

on a hill opposite (a, β), and the depression of the reflection of the bottom of the tower in the water (γ).

Show that the height of the tower is $2a \dfrac{\cos. \gamma}{\cos. a} \dfrac{\sin. (a-\beta)}{\sin. (\gamma-\beta)}$.

150. The shadows of two vertical walls, which are at right angles to each other, and are a feet and a' feet in height respectively, are observed when the sun is due south to be b feet and b' feet in breadth; show that if a be the sun's altitude, and β the inclination of the first wall to the meridian,

$$\cot. a = \sqrt{\frac{b^2}{a^2} + \frac{b'^2}{a'^2}}, \ \cot. \beta = \frac{a\,b'}{a'\,b}.$$

151. A flag-staff of known height h feet stands on a horizontal plane. The angle of elevation of its top is observed at three points, O, A, B, situated in a horizontal line through D, the foot of the flag-staff. If the increase in the angle of elevation in passing from O to A is equal to the increase in passing from A to B, and if OA $= a$ feet, AB $= b$ feet, and OD $= x$ feet, show that

$$(a - b)\,(x^2 + h^2) = a^2\,(2x - a - b).$$

152. A man walking due north along a straight road observes that at a certain milestone two distant objects bear N.E. and S.W., and that at the next milestone the straight line drawn to each object makes an angle of 30° with the direction of the road. Show that the distance to each object is $\sqrt{6}$ miles.

153. From a certain point on a level platform a flag-staff has the same bearing, S.W., and the same elevation, 45°, as a lamp-post of height of 14 feet. From another point, due west of the former point, and due north of the lamp-post, the elevation of the lamp-post is $\tan.^{-1} 1{\cdot}24$. Show that the height of the flag-staff is nearly 31 feet.

154. A man finds that he walks 40 feet in going straight down the slope of the embankment of a railway which runs due east and west, and when he has walked 20 feet along the foot of the embankment he finds that he is exactly N.E. of the point from which he started at the top of the bank. Show that the inclination of the bank to the horizon is 60°.

155. Three points, A, B, C, are situated in the same hori-

zontal line, and are such that AB = 150 feet, BC = 50 feet, and the angles of elevation of the top of a tower (which lies off the straight line) from the three points A, B, C are respectively 60°, 45°, 30°. Show that the height of the tower is 75 feet.

156. From a house on one side of a street observations (α, β) are made of the angle subtended by the house opposite, first from the level of the street, next from a room window at a known height, c, above the street. Prove that the height, h, of the opposite house is given by

$$\frac{c^2}{h^2} - \frac{c}{h} = \text{cot.}\ \beta\ \text{cot.}\ \alpha - \text{cot.}^2\ \alpha.$$

157. The hypothenuse of a right-angled triangle is a, and one of the acute angles is α. Prove that the distance between the centres of the squares described on the sides of the triangle, and external to it, is a sin. $(45° + \alpha)$.

158. From any point in the circumference of a circle three chords are drawn, such that the two outer ones are equally inclined to the inner one, and are together equal to it. Show that the outer extremities of the three chords are the angular points of an equilateral triangle.

159. Two triangles, ABC, A′BC, equal in every respect, are similarly situated on opposite sides of BC. Show that if A be 30°, and AA′ equal to BC, the values of B and C will be $75° \pm \frac{\alpha}{2}$, when cos. $\alpha = \frac{1 - \sqrt{3}}{2}$.

160. Two parallel chords of a circle, lying on the same side of a circle, subtend angles of 72° and 144° at the centre. Show that the distance between the chords is equal to half the radius of the circle.

161. A vessel starts from halfway between two points, B, C, and steams at uniform speed on a due north course at right angles to the line BC. An observer at B, at a certain moment, perceives that she is in the same straight line with a buoy A; five minutes later he sees that she bears N.E.; and five minutes later still an observer at C sees her to be in the same straight line with the buoy A. Prove that in the triangle ABC tan. A (tan. C − tan. B)^2 + 8 = 0.

162. The isosceles triangle ABC has each of its angles B, C

double of the angle A. If BD be drawn perpendicular to AC, and then DE be drawn perpendicular to BC, prove that

$$\text{CE} : \text{BE} :: \tan.^2 18^\circ : 1.$$

163. The angles at the base of a triangle are 22° 30′ and 112° 30′ respectively. Show that the area of the triangle is equal to the square on half the base.

164. In the ambiguous case for plane triangles, show that the rectangle contained by the third sides of the two triangles which satisfy the given conditions is equal to the difference of the squares of the two given sides.

165. If in the ambiguous case the area of the larger triangle is double that of the smaller, show that the tangent of one of the angles at the base is three times that of the other.

E.—Part I. Cap. XX.

166. A square and a regular hexagon are inscribed in the same circle. Show that their areas are as 4 to $3\sqrt{3}$.

167 A regular octagon has a side of 2 inches. Show that the radius of the inscribed circle is $(\sqrt{2} + 1)$ inches.

168. Show that in any plane triangle

(*a*) $\dfrac{2}{bc} + \dfrac{2}{ca} + \dfrac{2}{ab} = \dfrac{1}{\text{R}r}$.

(*b*) $r_a \cos.\dfrac{\text{A}}{2} = a \cos.\dfrac{\text{B}}{2}\cos.\dfrac{\text{C}}{2}$.

(*c*) $a \cos.\text{A} + b \cos.\text{B} + c \cos.\text{C} = 4\,\text{R} \sin.\text{A} \sin.\text{B} \sin \text{C}$.

169. In any plane triangle show that the sum of the products, taken two at a time, of the radii of the inscribed and escribed circles is $ab + bc + ca$.

170. Show that the radius of the circle which touches the sides AB, AC of the triangle ABC, and also touches the inscribed circle (of radius ρ), is

$$\frac{1 - \sin.\dfrac{\text{A}}{2}}{1 + \sin.\dfrac{\text{A}}{2}}\rho.$$

171. A circle is inscribed in the triangle ABC, and α, β, γ are the angles subtended at the centre by the sides of the triangle. Prove that

$$4 \sin. \alpha \sin. \beta \sin. \gamma = \sin. A + \sin. B + \sin. C.$$

172. In any plane triangle show that

$$r + r_b + r_c - r_a = 4R \cos. A.$$

173. If three circles, of radii r, r', r'', touch one another, prove that their common tangents at the points of contact meet at a point, and that the distance of this point from each point of contact $= \dfrac{\Delta}{r + r' + r''}$, where Δ is the area of the triangle whose vertices are at the centres of the three circles.

174. If the three bisectors of the angles of a triangle meet in O, show that the squares on OA, OB, OC are respectively proportional to

$$\frac{s-a}{a}, \; \frac{s-b}{b}, \; \frac{s-c}{c}.$$

175. Show that the sides of the triangle formed by the bisectors of the exterior angles of a triangle are $a \operatorname{cosec.} \frac{A}{2}$, $b \operatorname{cosec.} \frac{B}{2}$, $c \operatorname{cosec.} \frac{C}{2}$, and that its area is equal to $2.s.R$.

176. If the tangents at A, B, C to the circumscribed circle of a triangle ABC meet the opposite sides produced in the points D, E, F respectively, prove that the reciprocal of one of the distances AD, BE, CF is equal to the sum of the reciprocals of the other two.

177. If O is the centre of the circle described round an acute-angled triangle, and AO is produced to meet BC in D, show that

$$OD = \frac{R \cos. A}{\cos. (B - C)}.$$

178. If the inscribed circle of a triangle ABC touch the sides BC, CA, AB in D, E, F, prove that $\tan. ADB = \dfrac{2r_1}{b-c}$, where r_1 is the radius of that escribed circle which touches BC

179. Prove the following expression for the area of a triangle :—

$$\frac{2}{3} R^2 \{\sin.^3 A \cos. (B - C) + \sin.^3 B \cos. (C - A) + \sin.^3 C \cos. (A - B)\}.$$

180. The triangle ABC has a right angle at C; E is the point at which the inscribed circle touches BC, and F the point at which the circle drawn to touch AB and the sides AC, BC produced meets CA. Show that if EF be joined, the triangle FEC is half of the triangle ABC.

181. Show that the sum of the (n) perpendiculars from any point within a regular polygon upon the sides is constant, and equal to

$$n \frac{a}{2} \cot. \frac{\pi}{n}.$$

182. The radius of the circle inscribed in an isosceles triangle is 16 inches, and of the circle described about the triangle 50 inches. Show that the ratio of the base to one of the equal sides is as 8 to 5.

183. If the bisectors of the angles A, B, C of the triangle ABC meet the opposite sides in D, E, F, prove that

$$4 \text{ (area of ABC)} \times \text{(area of DEF)} = AD \,.\, BE \,.\, CF \,.\, r.$$

184. Show that the sum of the squares of the distances of the angular points of an equilaterial triangle from any point on the circumference of the inscribed circle is equal to five times the square on half the side of the triangle.

F.—Part I. Miscellaneous.

185. If $\sin. A \cos. B = \frac{1}{2}$, show that $\sec. 2A = 1 + \sec. 2B$.

186. If $\tan. C = \tan. A \tan. B$, show that

$$\tan. 2C = \{\sec. (A + B) + \sec. (A - B)\} \sin. A \sin. B.$$

187. If $2 \cos. \theta = x + \frac{1}{x}$, show that $2 \cos.^3 \theta = x^3 + \frac{1}{x^3}$.

188. If $\sin. 3x = - \sin. 2x$, prove that one value of $\cos. x$ is $\sin. 18°$.

189. If $A + B + C = 90°$, then

$$\frac{\cos. A + \sin. B + \sin. C}{\sin. A + \cos. B + \sin. C} = \frac{1 - \tan. \frac{A}{2}}{1 - \tan. \frac{B}{2}}.$$

190. If $A + B + C = 90°$, then

$$\sin. 2A + \sin. 2B + \sin. 2C = 4 \cos. A \cos. B \cos. C.$$

191. If $\cos.^3 2\theta + \cos.^2 2\theta + \mu^2 \cos. 2\theta = \mu^2$, show that $\mu \tan.^3 \theta + \tan.^2 \theta + \mu \tan. \theta = 1$.

192. Show that if $\cos. \theta = \frac{\cos. u - e}{1 - e \cos. u}$ then

$$\tan. \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}}.$$

193. If $\cos. A = x$, show that

$$\cos. 5A = 5x - 20x^3 + 16x^5$$

194. If $x \cos. \beta + y \sin. a = z$, and $x \sin. \beta - y \sin. a = 0$, prove that

$$\frac{x}{\sin. a} = \frac{y}{\sin. \beta} = \frac{z}{\sin. (a + \beta)}.$$

195. The sines of three acute angles in arithmetical progression are as the numbers 2, $\sqrt{6}$, $1 + \sqrt{3}$. Show that the angles are 45°, 60°, 75°.

196. If $\cos. \phi - \cos. \theta = m$, and $\sin. \phi - \sin. \theta = n$, show that the value of $\sin. (\theta + \phi)$ is $\frac{mn}{m^2 + n^2 - 1}$.

197. Show that

$$\tan.^{-1} \frac{x \cos. \phi}{1 - x \sin. \phi} - \tan.^{-1} \frac{x - \sin. \phi}{\cos. \phi} = \phi.$$

198. Show that sin. 63° is $\sqrt{2}$ times the arithmetic mean between sin. 18° and cos. 18°.

199. Show that $a \cos. \theta + b \sin. \theta$ will have its greatest value when $\theta = \tan.^{-1} \frac{b}{a}$.

200. A tower h feet high subtends an angle a at a person's eye; if the height of eye is a, show that x, the distance of the tower from the observer, is given by the equation

$$x^2 - h \cot. a \,.\, x + a^2 - ah = 0.$$

G.—Part II. Miscellaneous.

201. Prove that in any spherical triangle ABC,

$$\cos. b = \cos. (a + c) + 2 \sin. a \sin. c \cos.^2 \frac{B}{2}.$$

202. If the sum of the sides of a spherical triangle is 180°, show that the haversine of each angle is equal to the product of the cotangents of the two adjacent sides.

203. The equal sides, AB, AC, of an isosceles spherical triangle are bisected in P, Q respectively. Prove that

$$\sin. \frac{BC}{2} = 2 \sin. \frac{PQ}{2} \cos. \frac{AB}{2}.$$

204. If a be the side of an equilateral spherical triangle, and a' that of its polar triangle, show that

$$\cos. \tfrac{1}{2} a \cos. \tfrac{1}{2} a' = \tfrac{1}{2}.$$

205. If A be the angle of an equilateral spherical triangle, and A′ that of its polar triangle, show that

$$\text{hav. } A \,.\, \text{hav. } A' = \text{hav. } 60°.$$

206. Any point P is taken within a right-angled spherical triangle, and perpendiculars PM, PN are drawn to its sides CA, CB, which contain the right angle. Show that, if $CM = x$, $CN = y$, $PN = x'$, $PM = y'$,

$$\tan.^2 x + \tan.^2 y = \tan.^2 CP \quad \ldots \ldots \quad (i)$$

$$\sin. x = \frac{\sin. x'}{\cos. y'} \quad \ldots \ldots \quad (ii)$$

$$\sin. y = \frac{\sin. y'}{\cos. x'} \quad \ldots \ldots \quad (iii)$$

207. In a spherical triangle, right-angled at C, show that

(i) If $a + b + c = 90°$, $\tan. A = \text{cosec. } b - \sec. b$.

(ii) If $a + b + c = 180°$, $\cot. A = \tfrac{1}{2} (\sec. b - \cos. b)$.

208. If D be the middle point of the base BC of a spherical triangle ABC, and AB, AD, AC be in arithmetical progression, show that AB is a quadrant.

209. A spherical triangle ABC is right-angled at C. If p be the perpendicular from C upon AB, prove that

$$\sin.^2 p \sin.^2 c = \sin.^2 a + \sin.^2 b - \sin.^2 c.$$

210. In a spherical triangle ABC the arc AB is a quadrant, and CD is drawn perpendicular to AB. Prove that

$$\cot.^2 \mathrm{CD} = \cot.^2 \mathrm{A} + \cot.^2 \mathrm{B}.$$

211. In any spherical triangle prove that

$$\sin. \frac{a+b+c}{2} \sin. \frac{\mathrm{A}}{2} = \cos. \frac{\mathrm{B}}{2} \cos. \frac{\mathrm{C}}{2} \sin. a.$$

212. Through the vertical angle A of an isosceles spherical triangle there is drawn an arc of a great circle meeting the base in D. Show that

$$\tan. \frac{1}{2} \mathrm{BD} \tan. \frac{1}{2} \mathrm{CD} = \tan. \frac{1}{2} (\mathrm{BA} + \mathrm{AD}) \tan. \frac{1}{2} (\mathrm{BA} - \mathrm{AD}).$$

213. A triangle has two sides quadrants, and the difference between the sum of its angles and that of the angles of its polar triangle is a sixth of the sum of all six angles. Show that the angles of the triangle are 90°, 90°, 135°.

214. If the cosine of each of the equal sides of an isosceles spherical triangle is $\frac{1}{\sqrt{3}}$, and the base a quadrant, show that the angles of the triangle are 45°, 45°, 120°.

215. Show that the cosine of the angle between the chords of the two sides b, c of a spherical triangle is equal to

$$\sin. \frac{b}{2} \sin. \frac{c}{2} + \cos. \frac{b}{2} \cos. \frac{c}{2} \cos. \mathrm{A}.$$

216. If α, β, γ be the angles made with the sides BC, CA, AB of a spherical triangle by the connectors of their middle points with A, B, C respectively, then

$$\tan. \alpha \left(\frac{\cos. \frac{b}{2}}{\cos. \frac{c}{2}} - \frac{\cos. \frac{c}{2}}{\cos. \frac{b}{2}} \right) = \tan. \beta \left(\frac{\cos. \frac{c}{2}}{\cos. \frac{a}{2}} - \frac{\cos. \frac{a}{2}}{\cos. \frac{c}{2}} \right)$$

$$= \tan. \gamma \left(\frac{\cos. \frac{a}{2}}{\cos. \frac{b}{2}} - \frac{\cos. \frac{b}{2}}{\cos. \frac{a}{2}} \right).$$

217. The diagonals, AC, BD, of a spherical quadrilateral ABCD meet at right angles in O. If α, β, γ, δ are the arcs of great circles drawn from O perpendicular to the sides of ABCD, taken in order, prove that

$$\cot.^2 \alpha + \cot.^2 \gamma = \cot.^2 \beta + \cot.^2 \delta.$$

218. The square ABCD is inscribed in a sphere whose centre is O. If each side of the square is a, and the diameter of the sphere is d, prove that the cosine of the acute angle between the planes OAB and OBC is $\frac{a^2}{d^2 - a^2}$.

219. If in the spherical triangle ABC the arc which bisects the angle A meet the opposite side in D, and E be the middle point of that side, show that

$$\tan.\ DE \ .\ \tan.\ \frac{1}{2}(b + c) = \tan.\ \frac{a}{2} \tan.\ \frac{1}{2}(b - c).$$

220. A pyramid stands on an irregular polygonal base. If one face be an equilateral triangle, inclined at an angle of 45° to this base, show that the inclination of the next face, which is a right-angled isosceles triangle, is 60°.

221. If a, b, c, d be the sides of a spherical quadrilateral, taken in order, δ, δ' the diagonals, and ϕ the arc joining the middle points of the diagonals, show that

$$4 \cos.\ \frac{\delta}{2} \cos.\ \frac{\delta'}{2} \cos.\ \phi = \cos.\ a + \cos.\ b + \cos.\ c + \cos.\ d.$$

222. Prove that in a spherical triangle

$$\tan.\ c = \frac{\tan.\ b \cos.\ A + \tan.\ a \cos.\ B}{1 - \cos.\ A \cos.\ B \tan.\ a \tan.\ b}.$$

223. A spherical triangle ABC has all its angles right angles, and DEF is a great circle which meets BC in D, CA in E, and AB produced in F. Prove that

$$8 \cos.\ DE\ .\ \cos.\ EF\ .\ \cos.\ FD = \sin.\ 2BD\ .\ \sin.\ 2CE\ .\ \sin.\ 2AF.$$

224. The circumference of a great circle is trisected in A, B, C, and three equal small circles are described having A, B, C for their poles. Show that they will intersect one another if the area of each small circle is greater than three-fourths of the area of a great circle.

225. In a spherical triangle, right-angled at C, show that

$$\sin. \frac{1}{2} E = \frac{\sin. \frac{a}{2} \sin. \frac{b}{2}}{\cos. \frac{c}{2}}, \quad \cos. \frac{1}{2} E = \frac{\cos. \frac{a}{2} \cos. \frac{b}{2}}{\cos. \frac{c}{2}},$$

where $E = A + B + C - \pi$.

H.—Part III. Cap. I.–IV.

226. Find to five places of decimals the value of

$$\sqrt[8]{21 + \sqrt[6]{19}}.$$

Find the value of x in the following equations :—

227. $\dfrac{\text{vers. } 100^\circ}{\sqrt{x}} = \sqrt[3]{\text{vers. } 50^\circ}.$

228. $x^3 \cot. 108^\circ = 128 \sin. 72^\circ \cos. 18^\circ.$

229. $\dfrac{(66 \cdot 66)^{\frac{3}{4}} \sin.^2 33^\circ \cos.^{\frac{1}{2}} 337^\circ}{(\cdot 0033)^{\frac{2}{3}} x^{\frac{1}{3}}} = \tan.^5 57^\circ.$

230. $\dfrac{\cos. x}{\cos. (22^\circ + x)} = (2 \cdot 71828)^{\frac{3}{2}}.$

231. $\dfrac{27 \cdot 2968 \sin.^3 101^\circ}{\sqrt[3]{x}} = \dfrac{101 \cdot 78 \cos.^{\frac{3}{2}} 320^\circ}{(\cdot 009)^{\frac{1}{3}} \tan. 215^\circ}.$

232. $(\sin. 8^\circ + \cos. 8^\circ)^{2x} = 2 \sin. 16^\circ (\tan. 32^\circ)^x.$

233. $\log. x^2 + \log. 2x + 1 = 0.$

Solve the following plane triangles :—

234. $a = 32$, $b = 26$, $A = 46^\circ 20'$.

235. $a = 35$, $c = 20$, $A = 107^\circ 25'$.

236. $A = 84^\circ 52' 30''$, $B = 52^\circ 25' 30''$, $a = 235$.

237. $a = 56$, $b = 73$, $A = 41^\circ 10'$.

238. $b = 230$, $c = 327$, $A = 78^\circ 30'$.

239. $b = 3000$, $c = 3266$, $A = 49^\circ 28' 15''$.

240. $a = 93 \cdot 4$, $b = 70 \cdot 4$, $C = 116^\circ 12'$.

241. In the plane triangle ABC, $a = 152$ feet, $b = 188$ feet, $c = 142$ feet; find the angles A, B.

242. The sides of a triangle are 35, 43, 48. Show that one of the angles is 60°.

243. The sides of a plane triangle are ·75, ·93, 1·23; find the greatest angle.

244. In a plane triangle the sum of two sides is 160 feet, their difference is 35 feet, and the difference between the angles opposite these sides is 10°. Solve the triangle.

In the following triangles find the area:—

245. $a = 101{\cdot}5$ feet, $b = 167{\cdot}5$ feet, $C = 79° 33'$.

246. $b = 65{\cdot}3$ feet, $c = 89{\cdot}4$ feet, $C = 88° 30'$.

247. $a = 77$ feet, $b = 75$ feet, $c = 68$ feet.

248. In the triangle ABC, $a = 241$ yards, $b = 169$ yards, $C = 104° 3' 45''$; find the side of a square which has the same area.

249. Show that the area of a plane triangle, of which the sides are 98·29 yards, 105·72 yards, and 115·25 yards, is very nearly an acre.

250. Express in acres and decimals of an acre the area of a triangular field whose sides are respectively 316, 558, and 726 yards.

251. In a plane triangle ABC, AB = 37, BC = 45, AC = 74; find the length of the straight line drawn from C to the middle point of AB.

252. A quadrilateral figure ABPQ has one side, AB, 300 yards. The angle QAB = 58° 20′, PAQ = 37°, ABQ = 53° 30′, and PBQ = 45° 15′ Calculate the side PQ.

253. From a station, A, an object, O, bore S.E. by S., but after the observer had walked 1,200 yards S.W. the object bore E. by S. Find AO.

254. The angle of elevation of a tower 100 feet high from a place N.N.W. of it is 45°, and from another place E. by S. of it is 60°; find the bearing of the second place from the first.

255. What angle will a flag-staff 18 feet high, on the top of a tower 200 feet high, subtend to an observer on the same level with the foot of the tower, and 100 yards distant from it?

256. A quadrilateral has its sides AB, BC, CD, DA and its diagonal BD, respectively equal to 25, 39, 52, 60, and 65 inches. Show that the angles at A and C are right angles, and find the other two angles of the quadrilateral.

257. The angle of elevation of the top of a hillock being 2° 33′ 15″, a man proceeds to walk up it (by a direct course),

the angle of elevation of his path for 100 yards being 2°, and afterwards 3°. Find the vertical height of the hillock.

258. A flag-staff, 27 feet in height, standing on the edge of a cliff, subtends an angle of 0° 40′ at a ship at sea, the angle of elevation of the cliff being 15°. Find the distance of the cliff from the ship (the point of observation being considered to be in the same horizontal plane with the foot of the cliff).

259. Two objects, A and B, lie in the same straight line and the same horizontal plane with the base, D, of a tower CD, whose height is 98 feet. If the angles ACB, ABC are observed to be 7° 21′ and 8° 12′ respectively, find the distance AB.

260. At a station due south of a circular fort a man observes the horizontal angle subtended by the fort to be 2° 11′ 30″. He then walks E.N.E. to a station a quarter of a mile distant, and finds the angle subtended to be the same as before. Find the diameter and distance of the fort.

261. A flag-staff which leans over to the east is found to cast shadows of 198 feet and 202 feet, when the sun is due east and due west respectively, and his altitude is 7°. Find the length of the flag-staff and its inclination to the vertical.

262. At two points in a straight sea-wall, 440 yards apart, the lines drawn to a ship are found to be inclined respectively at angles 60° and 66° to the direction of the wall. Find the distance of the ship from the nearest point of the wall. (N.B.—The stations lie on opposite sides of the ship.)

263. The elevations of a tower from two points in the same straight line with its foot, at a distance of 41 yards from one another, are 58° 23′ 30″ and 27° 26′ 15″. Find the height of the tower.

264. A privateer is lying 10 miles W.S.W. of a harbour, when a merchantman leaves it steering S.E. 8 miles an hour. If the privateer overtakes the merchantman in 2 hours, find her course and rate of sailing.

265. At the point C is a light-ship. The directions CA, CB are known to include an angle of 30° 20′, and A bears N.E. by E. from C. The distance CA being 3 miles, a steamer is seen to start from A at a speed of 9 knots. In what direction must she steer so that in 40 minutes she may appear to the light-ship in the line CB ?

266. The legs of a pair of compasses are 10·5 inches and 6 inches respectively. At what angle must they be placed to mark off a distance of 12·5 inches?

267. An observer, situated due south of a landmark A, notices that the bearing of another landmark, B, is N. 15° 22′ W. He then walks one statute mile N. 45° W., and finds that B is due east while A bears S. 84° 15′ E. Find the distance AB.

268. A man walking along a road which runs N.E. observes that a distant object bears N. 60° E. After walking 1,500 yards further the object bears N. 80° E. Find the distance of the object from the first point of observation, and its shortest distance from the road.

269. From the junction of two lines of railway two trains start together at different speeds. The angle between the directions of the lines is 108°, and the trains reach their respective destinations, which are 70 miles apart, in $1^h\ 30^m$. If the speed of one train be 20 miles an hour, find the speed of the other.

270. At $4^h\ 30^m$ P.M. on a certain day a vessel sailing S. by W. at a uniform rate of 7·5 knots passed another vessel sailing E. by S.$\frac{1}{2}$S. 9 knots. Assuming the courses and rates to have been uniform, find the bearing and distance of one ship from the other at the preceding noon.

271. A column stands at the top of a hill whose inclination is 10°, and at two stations 40 feet apart on the slope of the hill, and in a direct line with the foot of the column, the angles of elevation (above the horizon) of the top of the column are 40° and 46°. Find its height.

272. A man standing at a certain point in a straight road observes that the straight lines drawn from that point to two landmarks on the same side of the road are each inclined at an angle of 45° to the direction of the road. He walks to a point 250 yards further along the road, and there finds that the former angles of inclination are changed to 25° and 65° respectively. Find the distance between the landmarks.

273. A hill rises from a horizontal plain. On the hill is a tower. From a point C in the plain the angles of elevation of the top, A, and the bottom, B, of the tower are respectively

72° 15′ and 54° 45′. A distance CE of 200 feet is then measured horizontally, and the angle of elevation of A as seen from E is then 44° 12′. The straight lines CA, EA, CB being in the same vertical plane, find the height AB.

274. A spectator observes the explosion of a meteor due south of him at an altitude of 28° 45′. To another spectator 11 miles S.S.W. of the former it appears at the same instant to have an altitude of 42° 15′ 30″. Show that there are two possible heights above the earth's surface at which it may have exploded, and find these heights.

275. A person is standing on a cliff looking north (200 feet above the sea), and observes the elevation of a cloud to be 53° 7′ 45″, and the depression of its shadow on the sea to be 7° 7′ 30″. The sun being due south, and the line joining sun and cloud being inclined at an angle of 45° to the horizon, find the height of the cloud above the sea.

276. A tower 120 feet high stands in the middle of a horizontal field whose shape is that of an equilateral triangle. From the top of the tower each side of the field subtends an angle of 100°. Find the length of a side of the field.

277. Two towers on a hill-side are 70 feet and 130 feet high respectively. The line connecting their bases, which are 80 feet apart, makes an angle of 10° with the horizon. Find the inclination to the horizon of the line joining their tops.

278. A ship sailing at a uniform rate was observed to bear N. 30° 57′ 30″ E. After 20 minutes she bore N. 35° 32′ 15″ E., and after 10 minutes more N. 37° 52′ 30″ E. Find the direction in which she was sailing.

K.—Part III. Cap. V.–VII.

In the following spherical triangles find the angles:—

279. $a = 158° 54' 15''$, $b = 118° 21'$, $c = 63° 42'$.

280. $a = 95° 18' 15''$, $b = 50° 45' 15''$, $c = 69° 12' 45''$.

281. $a = 58° 34' 30''$, $b = 60° 18'$, $c = 92° 10'$.

In the following spherical triangles find the third side:—

282. $a = 112° 30'$, $b = 53° 45' 30''$, $C = 23°$; find c.

283. $b = 62° \ 36' \ 45''$, $c = 100° \ 10' \ 15''$, $A = 81° \ 24'$; find a.

284. $a = 107° \ 15'$, $b = 94° \ 12' \ 45''$, $C = 91° \ 14' \ 30''$; find c.

285. $b = 70° \ 14' \ 15''$, $c = 38° \ 46' \ 15''$, $A = 48° \ 56'$; find a.

286. $b = 58° \ 25'$, $c = 49° \ 10'$, $A = 71° \ 18' \ 30''$; find a.

In the following spherical triangles find the other angles :—

287. $a = 65° \ 20' \ 15''$, $b = 50° \ 30' \ 15''$, $C = 118° \ 10' \ 45''$.

288. $a = 49° \ 10'$, $b = 58° \ 25'$, $C = 71° \ 18' \ 30''$.

289. $a = 117° \ 30'$, $b = 57° \ 45'$, $C = 84° \ 15'$.

290. $a = 45° \ 30'$, $b = 75° \ 45'$, $C = 80°$.

291. In the spherical triangle ABC, given $A = 124° \ 13'$, $B = 49° \ 7'$, $a = 115° \ 6'$; find b.

292. Having given $b = 64° \ 30' \ 45''$, $c = 95° \ 7' \ 45''$, $C = 100° \ 48', 15''$; find a.

293. Having given $A = 126° \ 37'$, $B = 48° \ 30'$, $a = 115° \ 20'$; solve the triangle.

294. Having given $a = 101° \ 37'$, $A = 98° \ 13' \ 30''$, $B = 85° \ 17'$; find b.

295. Having given $a = 52° \ 13'$, $b = 70° \ 21'$, $A = 46° \ 18'$; find B.

296. Having given $A = 67°$, $B = 120°$, $c = 43°$; find C.

297. Having given $A = 71°$, $B = 59°$, $c = 38°$; find C.

298. Having given $A = 112° \ 25'$, $B = 73° \ 15'$, $c = 24° \ 38'$; find a, b, C.

299. Having given $B = 62° \ 20' \ 15''$, $C = 60° \ 13' \ 45''$, $a = 150° \ 10' \ 30''$; find A.

300. Having given $A = 130° \ 50'$, $B = 121° \ 35'$, $c = 108° \ 41' \ 30''$; find C.

301. Having given $A = 85°$, $B = 65°$, $c = 130°$; find a, b.

302. Having given $A = 110°$, $B = 54° \ 30'$, $c = 22° \ 30' \ 30''$; find C.

303. Having given $A = 105° \ 15' \ 30''$, $B = 65° \ 30'$, $C = 85° \ 20' \ 45''$; find the sides.

304. Having given $A = 121° \ 36' \ 30''$, $B = 42° \ 15' \ 15''$, $C = 34° \ 15'$; find the sides.

305. Having given $A = 94° 30'$, $B = 104° 33' 45''$, $C = 144° 9' 30''$; find a.

306. Find the side of a spherical triangle each angle of which is $70° 46' 15''$.

Solve the following right-angled spherical triangles :—

307. $C = 90°$, $a = 49° 17' 30''$, $b = 95° 36' 15''$.

308. $C = 90°$, $a = 72° 27'$, $c = 91° 18'$.

309. $C = 90°$, $a = 54° 41' 45''$, $B = 101° 48'$.

310. $C = 90°$, $a = 18° 25' 45''$, $b = 72° 15' 30''$.

311. $C = 90°$, $a = 24° 15'$, $c = 145° 30'$.

312. $C = 90°$, $a = 132° 39' 30''$, $B = 78° 10'$.

313. $A = 90°$, $a = 98° 14' 30''$, $b = 54° 41' 45''$.

314. $C = 90°$, $a = 57° 16'$, $b = 96° 24' 15''$.

315. $C = 90°$, $a = 22° 30'$, $A = 30°$.

316. $C = 90°$, $a = 38° 47' 15''$, $A = 42° 50' 45''$.

317. $C = 90°$, $a = 37° 25' 30''$, $B = 40° 4' 15''$.

318. $A = 90°$, $B = 101° 50'$, $C = 48° 27' 15''$.

319. $c = 90°$, $A = 131° 30'$, $B = 120° 32'$.

320. $a = 90°$, $A = 100°$, $B = 74° 36' 30''$.

321. $a = 90°$, $A = 80°$, $B = 75°$.

322. $a = 90°$, $A = 85°$, $b = 97°$.

323. $a = 90°$, $B = 79° 54' 15''$, $C = 97° 56' 45''$

324. $a = 90°$, $b = 78° 14' 30''$, $c = 50° 10'$.

325. In the spherical triangle ABC, $a = 90°$, $b = 90°$, and c is one half the side of an equilateral triangle in which each angle is $75°$. Find A, B and C.

ANSWERS TO APPENDIX.

A.

1. $220\sqrt{2}$ yards.
2. 8 feet 11·4 inches.
3. 12 feet 9·6 inches.
4. $4° 17' 50''$.
5. $25° 42' 51''$; ·449.
7. (*a*) 1910 miles nearly; (*b*) 30 feet.
8. ·1744 inch.
9. 92,678,723 miles.
10. $\frac{1}{60}$; $57' 18''$ nearly.
11. 90°.
12. 141°·43 nearly.
13. $65° 28' 51''$.
14. $\frac{1}{4}$.
15. 1·37 nearly.
16. 360°.
17. 1·91 mile nearly.
18. $114° 35' 30''$.
19. $\frac{180}{\pi}$.
21. 75°, 30°.
22. 27°, 69°, 111°, 153°.
23. $48° 11' 30''$; ·841.

B.

80. $\frac{3}{5}$ or $\frac{4}{5}$.
82. $\frac{1}{3}$.
83. $-\frac{333}{644}$
84. $\frac{323}{325}$.
85. $\frac{1}{\sqrt{2}}$.
87. $\frac{5}{12}$.
89. $-\sqrt{\frac{1}{10}}$.
90. sine $\frac{11}{\sqrt{130}}$, tangent $-\frac{11}{3}$.

C.

93. $-\frac{4}{3}$.
94. $\frac{5}{12}$.
95. tan. $\theta = \sqrt{2}$; tan. $3\theta = -\frac{1}{5}\sqrt{2}$.
96. $\frac{1}{2}$ or -1.
97. $-\frac{\sqrt{3}}{2}$.
98. (*a*) $\theta = n\pi$ or $n\pi + (-1)^n\frac{\pi}{6}$.
(*b*) $\theta = 2n\pi \pm \frac{\pi}{3}$.
(*c*) $\theta = n\pi$ or $n\pi - \frac{\pi}{4}$.
(*d*) $\theta = 2n\pi \pm \frac{2\pi}{3}$.
(*e*) $\theta = n\pi$ or $n\pi \pm \frac{\pi}{4}$.

100. $2x = n\pi \pm \frac{\pi}{3}$. 110. $\frac{3}{2}$. 111. 1·58496.

112. 4·48289 nearly; 2·443 nearly. 113. 1·062 nearly.

H.

226. 1·47687. 227. 2·73605. 228. – 7·0895.
229. 41411. 230. 61° 59′. 231. ·00016736.
232. – ·8345 nearly. 233. ·3684.
234. B = 35° 59′ 45″, C = 97° 40′ 15″, c = 43·84.
235. B = 39° 32′ 30″, C = 33° 2′ 30″, b = 23·36.
236. C = 42° 42′, b = 187, c = 160.
237. B = 59° 6′ or 120° 54′, C = 79° 44′ or 17° 56′, c = 83·71 or 26·2.
238. B = 38° 43′, C = 62° 47′, a = 360·3.
239. B = 60°, C = 70° 31′ 45″, a = 2633.
240. A = 36° 53′ 45″, B = 26° 54′ 15″, c = 139·6.
241. A = 52° 38′, B = 79° 25′ 15″. 243. 93° 30′ 45″.
244. Angles 26° 48′, 16° 48′, 136° 24′. Sides 97·5, 62·5, 149·1.
245. 8359·6 square feet. 246. 2049·5 square feet.
247. 2310 square feet. 248. 140·5 yards.
250. 17·2046 acres. 251. 58·38.
252. 206 yards nearly. 253. 1411 yards.
254. S. 42° 29′ E. 255. 2° 13′ 45″.
256. B = 120° 30′ 30″, D = 59° 29′ 30″.
257. 30 feet nearly. 258. 2159·2 feet.
259. 327·9 feet.
260. Diameter 22 yards; distance 575 yards.
261. 24·6 feet; inclination to vertical 4° 39′.
262. 430·3 yards. 263. 31·28 yards.
264. S. 70° E. 10·9 knots. 265. S. 78° 48′ E.
266. 94° 33′. 267. 364 yards nearly.
268. 2515·6 yards; 651·6 yards. 269. 36·43 miles per hour.
270. S. 64° 48′ W. 50 miles nearly.
271. 114·2 feet. 272. 731 yards.
273. 154·5 feet. 274. 4·33 miles or 13·21 miles.
275. 1000 feet. 276. 394 feet.
277. 43° 9′ 45″. 278. S. 44° 38′ E.

K.

279. A = 156° 19′, B = 100° 59′ 45″, C = 90°.
280. A = 115° 58′ 15″, B = 44° 22′ 15″, C = 57° 35′.
281. A = 51° 31′ 15″, B = 52° 50′, C = 113° 33′ 30″.
282. c = 62° 38′. 283. a = 87° 10′ 15″.
284. c = 89° 56′ 15″. 285. a = 49° 24′ 15″
286. a = 56° 42′. 287. A = 53° 24′, B = 42° 59′.

288. $A = 59^\circ\, 2'\, 15''$, $B = 74^\circ\, 53'\, 45''$.

289. $A = 116^\circ\, 23'\, 30''$, $B = 58^\circ\, 39'\, 30''$.

290. $A = 47^\circ\, 16'\, 15''$, $B = 86^\circ\, 32'\, 45''$.

291. $b = 55^\circ\, 53'\, 15''$. 292. $a = 79^\circ\, 41'$.

293. $b = 57^\circ\, 30'$, $c = 82^\circ\, 26'\, 30''$, $C = 61^\circ\, 41'\, 30\, '$.

294. $b = 80^\circ\, 31'\, 15''$ or $99^\circ\, 28'\, 45''$.

295. $B = 59^\circ\, 29'$ or $120^\circ\, 31'$. 296. $C = 38^\circ\, 53'\, 15''$.

297. $C = 61^\circ\, 54'\, 15''$.

298. $a = 107^\circ\, 42'\, 15''$, $b = 99^\circ\, 19'\, 15''$, $C = 23^\circ\, 51'\, 30''$.

299. $A = 153^\circ\, 50'$. 300. $C = 123^\circ\, 18'$.

301. $a = 104^\circ\, 5'\, 45''$, $b = 61^\circ\, 55'\, 45''$.

302. $C = 25^\circ\, 8'$.

303. $a = 104^\circ\, 39'\, 30''$, $b = 65^\circ\, 51'\, 15''$, $c = 91\ \ 49'\, 15''$.

304. $a = 76^\circ\, 36'$, $b = 50^\circ\, 11'\, 15''$, $c = 40^\circ$.

305. $a = 77^\circ\, 13'\, 30''$. 306. $60^\circ\, 34'\, 30''$.

307. $c = 93^\circ\, 39'\, 15''$, $B = 94^\circ\, 15'\, 15''$, $A = 49^\circ\, 25'\, 45''$.

308. $A = 72^\circ\, 29'\, 45''$, $B = 94^\circ\, 7'$, $b = 94^\circ\, 19'$.

309. $A = 55^\circ\, 23'$, $b = 104^\circ\, 21'\, 30''$, $c = 98^\circ\, 14'\, 30''$.

310. $A = 19^\circ\, 17'$, $B = 84^\circ\, 13'\, 30''$, $c = 73^\circ\, 11'\, 45''$.

311. $A = 46^\circ\, 28'\, 45''$, $b = 154^\circ\, 40'\, 30''$, $B = 130^\circ\, 57'$.

312. $A = 131^\circ\, 32'\, 45''$, $b = 74^\circ\, 5'\, 45''$, $c = 100^\circ\, 42'$.

313. $B = 55^\circ\, 33'$, $C = 101^\circ\, 48'\, 15''$, $c = 104^\circ\, 21'\, 45''$.

314. $A = 57^\circ\, 2\ \ \ 45''$, $B = 95^\circ\, 23'\, 30''$, $c = 93^\circ\, 27'\, 30''$.

315. $b = 45^\circ\, 50'\, 45''$ or $134^\circ\, 9'\, 15''$, $B = 69^\circ\, 37'$ or $110^\circ\, 23'$, $c = 49^\circ\, 56'\, 30''$ or $130^\circ\, 3'\, 30''$.

316. $b = 60^\circ\, 3'\, 15''$ or $119^\circ\, 56'\, 45''$, $B = 70^\circ\, 9'\, 15''$ or $109^\circ\, 50'\, 45''$, $c = 67^\circ\, 6'$ or $112^\circ\, 54'$.

317. $b = 27^\circ\, 4'\, 30''$, $c = 45^\circ$, $A = 59^\circ\, 15'\, 15''$.

318. $a = 100^\circ\, 42'$, $b = 105^\circ\, 54'\, 15''$, $c = 47^\circ\, 20'\, 30''$.

319. $a = 127^\circ\, 18'\, 30''$, $b = 113^\circ\, 50'$, $C = 109^\circ\, 40'$.

320. $b = 78^\circ\, 14'\, 15''$, $c = 50^\circ\, 10'\, 15''$, $C = 49^\circ\, 8'\, 15$.

321. $b = 78^\circ\, 45'\, 45''$, $c = 131^\circ\, 9'\, 15''$, $C = 132^\circ\, 8'\, 15'$.

322. $c = 54^\circ\, 38'$, $B = 98^\circ\, 35'\, 45''$, $C = 54^\circ\, 19'\, 30''$.

323. $b = 80^\circ$, $c = 97^\circ\, 49'\, 30''$, $A = 88^\circ\, 36'\, 45''$.

324. $A = 100^\circ$, $B = 74^\circ\, 36'\, 15''$, $C = 49^\circ\, 8'$.

325. $A = B = 90^\circ$; $C = c = 34^\circ\, 47'$.

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